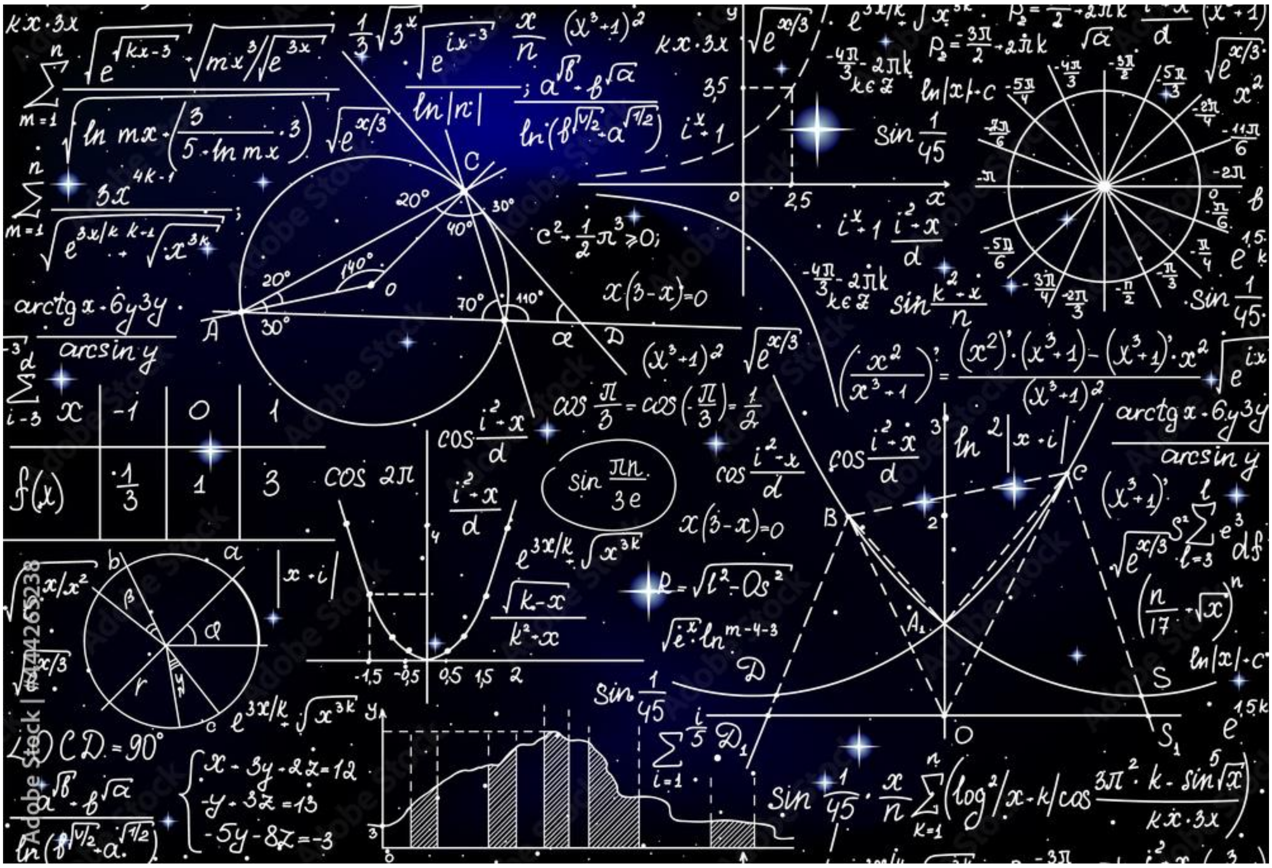
C.A.P.E.

PURE MATHEMATICS UNIT 1

CRASH COURSE

WORKBOOK



NICHOLAS PAUL

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Module 1:

Basic algebra and Functions

REMAINDER AND FACTOR THEOREM

- Review Question 1

$x + 1 \overline{)3x^3 - 2x^2 + 4x + 1}$

- Review Question 2

$x - 1 \overline{)7x^3 - 4x^2 + 2x - 5}$

- Past Paper Question – 2015 Paper 2 ROR #1 b)

The polynomial $f(x) = x^3 + px^2 - x + q$ has a factor $(x - 5)$ and a remainder of 24 when divided by $(x - 1)$.

(i) Find the values of p and q. (4 marks)

(ii) Hence, factorize $f(x) = x^3 + px^2 - x + q$ completely. (5 marks)

Hint: Obtain 2 equations and solve simultaneously

- Past Paper Question – 2016 Paper 2 #1 a)

Let $f(x) = 2x^3 - x^2 + px + q$.

(i) Given that $x + 3$ is a factor of $f(x)$ and that there is a remainder of 10, when $f(x)$ is divided by $x + 1$ show that $p = -25$ and $q = -12$. (7 marks)

(ii) Hence, solve the equation $f(x) = 0$. (6 marks)

LOGIC AND REASONING I

- Practice Question 1

  Construct a truth table for $\sim\mathbf{p} \wedge \sim\mathbf{q}$.

- Practice Question 2

  Construct a truth table for $(\mathbf{p} \wedge \mathbf{q}) \vee (\mathbf{q} \vee \mathbf{r})$.

- Practice Question 3

  If we have the conditional statement, $\sim\mathbf{p} \rightarrow \mathbf{q}$ then:

  - Converse:
  - Inverse:
  - Contrapositive:

- Past Paper Question – 2018 Paper 2 #1 a)

(i) Let **p** and **q** be any two propositions. Complete the truth table below. (4 marks)

p	**q**	**~p**	**~q**	**(p ∨ q)**	**~(p ∨ q)**	**~p ∧ ~q**
T	T					
T	F					
F	T					
F	F					

(ii) Hence, state whether the statements ~(**p** ∨ **q**) and ~**p** ∧ ~**q** are logically equivalent. **Justify your response.** (1 mark)

LOGIC AND REASONING II

- Past Paper Question – 2022 Paper 2 #1 a)

(i) Let **p** and **q** be any two propositions. Complete the truth table below. (3 marks)

p	**q**	**~q**	(**~q** ∧ **p**)	**p** ∨ (**~q** ∧ **p**)
T	T			
T	F			
F	T			
F	F			

(ii) Hence, state whether the statements (**~q** ∧ **p**) and **p** ∨ (**~q** ∧ **p**) are logically equivalent. **Justify your response.** (2 marks)

- Practice Question 1

Prove (**p** ∨ **q**) ∧ **~p** ≡ **~p** ∧ **q**.

- Practice Question 2

  Prove $\mathbf{p} \wedge (\sim\mathbf{p} \vee \mathbf{q}) \equiv \mathbf{p} \wedge \mathbf{q}$.

- Practice Question 3

  Prove $\sim(\mathbf{p} \wedge \sim\mathbf{q}) \equiv \sim\mathbf{p} \vee \mathbf{q}$.

- Practice Question 4

  Prove $[\mathbf{p} \wedge (\sim\mathbf{p} \vee \mathbf{q})] \vee \mathbf{q} \equiv \mathbf{q}$.

- Practice Question 5

  Prove $\sim(\mathbf{p} \wedge \mathbf{q}) \wedge (\sim\mathbf{p} \vee \mathbf{q}) \equiv \sim\mathbf{p}$.

- Practice Question 6

  Prove $\sim(\mathbf{p} \wedge \sim\mathbf{q}) \wedge (\sim\mathbf{p} \vee \sim\mathbf{q}) \equiv \sim\mathbf{p}$.

FIELD AXIOMS AND METHODS OF PROOF

- Practice Question 1

*	a	b	c	d
a	d	a	b	c
b	a	b	c	d
c	b	c	d	a
d	c	d	a	b

Given that the identity element, e = b.

Find the Inverse for the following, the first one has been done for you.

(i) a * c = b

(ii) b *____ = b

(iii) c *____ = b

(iv) d *____ = b

- Practice Question 2

*	p	q	r	s
p	s	p	q	r
q	p	q	r	s
r	q	r	s	p
s	r	s	p	q

(i) Determine if the above matrix is commutative.

(ii) Determine the identity element, e.

p *____ = p

q *____ = q

r *____ = r

s *____ = s

Therefore,

the identity element, e = _____

(iii) Determine the inverse for the following:

p *____ = e

q *____ = e

r *____ = e

s *____ = e

- Practice Question 3

  Given that $x, y \in \mathbb{Z}$. Determine if $*$ is closed in $\boldsymbol{x * y = \frac{x}{y}}$.

- Practice Question 4

  Prove that the sum of two (2) even numbers is even.

- Practice Question 5

  Prove that the sum of two (2) odd numbers is always even.

- Practice Question 6

  Given that $x > y$ and $k < 0$. Show that $kx < ky$.

  Hint: When an inequality is multiplied by a negative, the sign flips.

- Practice Question 7

  Given that $x < y$ and $k > 0$. Prove that $kx < ky$.

- Practice Question 8

  Using P.M.I. show that $P_n: 6^n - 1$ is divisible by 5, $\forall\, n \in \mathbb{N}$.

PROOF BY MATHEMATICAL INDUCTION I

- Practice Question 1

  Using P.M.I. show that $3^{4n} - 1$ is divisible by 16, $\forall\, n \geq 1$.

- Practice Question 2

  Using P.M.I. show that $\sum_{r=1}^{n} r = \frac{1}{2}(n)(n+1)$, $\forall\, n \geq 1$.

- Practice Question 3

  Using P.M.I. show that $\sum_{r=1}^{n} (4r-3) = n(2n-1)$, $\forall\, n \in \mathbb{N}$.

- Practice Question 4

  Using P.M.I. show that $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$, $\forall\ n \in \mathbb{N}$.

PROOF BY MATHEMATICAL INDUCTION II

- Past Paper Question – 2017 Paper 2

  Use mathematical induction to prove that,

  $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{n.(n+1)}=\frac{n}{n+1}$ for $n \in \mathbb{N}$.

- Practice Question 1

  Using P.M.I. show that $2^n < n$,$\forall\, n \in \mathbb{N}$.

- Practice Question 2

  Using P.M.I. show that $2^n > n^2 - 1$,$\forall\, n \geq 4$.

INDICES AND SURDS

- Practice Question 1

  Simplify $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} \times \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

- Practice Question 2

  Simplify

  $$\frac{3\sqrt{5}+2\sqrt{3}}{2\sqrt{5}-4\sqrt{3}}$$

- Past Paper Question – 2008 Paper 2 ROR #1 b) (ii)

  Without using calculators or tables, evaluate

  $$\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}+\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}$$ (5 marks)

- Past Paper Question – 2005 Paper 2 #3 a)

  Show

  $$\sqrt{11}+\sqrt{7}=\frac{4}{\sqrt{11}-\sqrt{7}}$$

- Past Paper Question – 2004 Paper 2 #3

(i) Show

$$\frac{4}{2^x} = 2^{2-x}$$

(ii) Solve $2^x + 2^{2-x} = 5$

Hint: Disguise Quadratic

- Past Paper Question – 2008 Paper 2 TT #2 a)

  Without using calculators or tables, evaluate,

$$\sqrt{\frac{27^{10} + 9^{10}}{27^{4} + 9^{11}}}$$

(8 marks)

- Past Paper Question – 2015 Paper 2 ROR #2 b) (i)

  Solve the following equation:

$$3 - \frac{4}{(9)^x} - \frac{4}{(81)^x} = 0$$

(7 marks)

LOGARITHMS

- Practice Question 1

  $\ln x^2 - \ln x$

- Past Paper Question – 2008 Paper 2 ROR #2 b) (iii)

  Without the use of calculators or tables, evaluate

$$\log_{10}\left(\frac{1}{2}\right) + \log_{10}\left(\frac{2}{3}\right) + \log_{10}\left(\frac{3}{4}\right) + \cdots + \log_{10}\left(\frac{8}{9}\right) + \log_{10}\left(\frac{9}{10}\right).$$

(3 marks)

- Past Paper Question – 2009 Paper 2 #1 c)

  Solve the equation $\log_4 x = 1 + \log_2 2x\ , x > 0$. (8 marks)

- Past Paper Question – 2012 Paper 2 #2 c)

  Without the use of calculators or tables, evaluate

  (i) $\log_{10}\left(\frac{1}{3}\right) + \log_{10}\left(\frac{3}{5}\right) + \log_{10}\left(\frac{5}{7}\right) + \log_{10}\left(\frac{7}{9}\right) + \log_{10}\left(\frac{9}{10}\right)$. (3 marks)

  (ii) $\sum_{r=1}^{99} \log_{10}\left(\frac{r}{r+1}\right)$. (4 marks)

- Past Paper Question – 2014 Paper 2 #2 c)

  Solve EACH of the following equations:

  (i) $e^x + \frac{1}{e^x} - 2 = 0$ (4 marks)

(ii) $\log_2(x+1) - \log_2(3x+1) = 2$ (4 marks)

- Past Paper Question – 2014 Paper 2 #2 b)

  Given that $a^3 + b^3 + 3a^2b = 5ab^2$, show that $3 \log \left(\frac{a+b}{2}\right) = \log a + 2 \log b$.

  (5 marks)

MODELLING AND SKETCHING GRAPHS

- Practice Question 1

  A mass in grams (g) of a piece of radioactive material at time (t years) is modelled by the following formula:

  $$m(t) = 6e^{-0.12t}$$

  Given that the present mass of the radioactive material i.e. when $t = 0$ is 6 g.

  Determine:

  (i) The mass present after 5 years.

  (ii) Time taken for the mass to reach half of the original value.

(iii) Time taken for the mass to reach $2\ g$.

- Practice Question 2

A mass in grams (g) of a piece of radioactive material at time (t years) is modelled by the following formula:

$$m(t) = 250(\frac{1}{2})^t$$

Given that the present mass of the radioactive material i.e., when $t = 0$ is $6\ g$.

Determine:

(i) The present mass of the radioactive material.

(ii) The mass present after 5 years.

(iii) Time taken for the mass to reach $125\ g$.

- Practice Question 3

  Value, v - value in thousands

  Time, t - time in years

  Given that the above is modelled by the formula :

$$V(t) = 365e^{-0.15t}$$

Determine:

(i) The initial value

(ii) Time taken to reach 91.25 thousand dollars.

- Practice Question 4

  Sketch the transformation for $y = (x - 2)^2 - 5$.

  Hint: The parent function is $y = x^2$.

- Practice Question 5

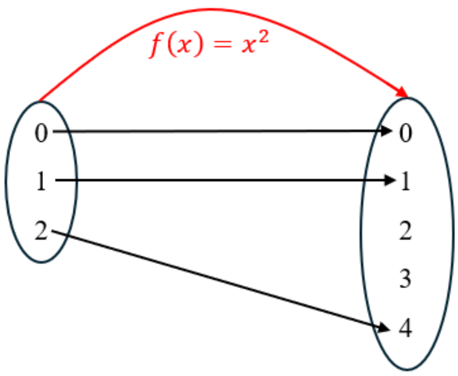
  Sketch the transformation for $y = \frac{1}{x-3} + 1$.

- Practice Question 6

  Sketch the transformation for $y = \frac{3x+7}{x+2}$.

FUNCTIONS I

- Practice Question 1



In the above relation diagram, determine the following:

(i) Domain:

(ii) Co-domain:

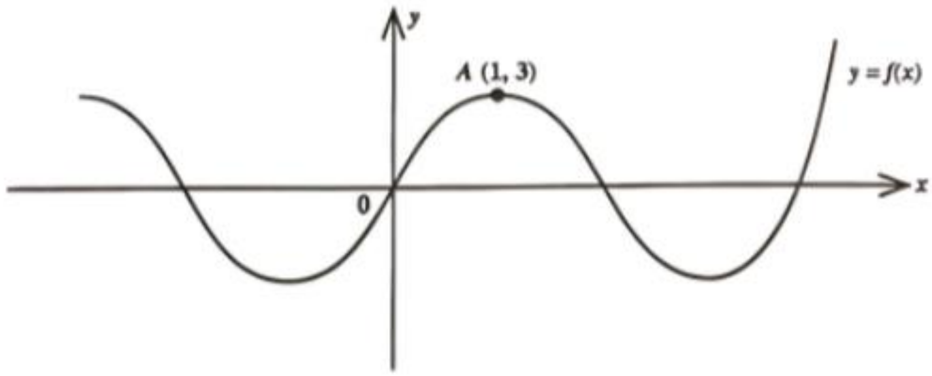
(iii) Range:

- Practice Question 2

Determine if $f(x) = x^2 + 1$ is injective such that $x \in \mathbb{R}^+$.

- Practice Question 3

(a) The diagram below (not drawn to scale) shows the graph $y = f(x)$ which has a local maximum point at A (1,3).



Determine the coordinates of the maximum point on each of the following graphs.

(i) $y = f(x) - 2$

(ii) $y = f(x + 3)$

(b) The function, f, is defined on $\mathbb{R}$ by $f: x \rightarrow 3x - 2$

(i) Show that f is one-to-one.

(ii) Hence, or otherwise, find the value of $x \in \mathbb{R}$ for which

$f(f(x+3)) \equiv f(x-3)$

- Practice Question 4

Determine if $f(x) = \frac{x+3}{x-1}$ is one-to-one given that $x \neq 1$.

FUNCTIONS II

- Practice Question 1

  Find the inverse of $f(x) = \frac{3x-1}{2}$.

- Past Paper Question – 2014 Paper 2 #2 a)

  The functions f and g are defined as follows:

  $f(x) = 2x^2 + 1$

  where $1 \leq x < \infty, x \in \mathbb{R}$.

  $g(x) = \sqrt{\frac{x-1}{2}}$

(i) Determine, in terms of x,

a) $f^2(x)$ (3 marks)

b) $f[g(x)]$ (3 marks)

(ii) Hence, or otherwise, state the relationship between f and g. (1 mark)

- Past Paper Question – 2013 Paper 2 #2 a) b)

  (a) Let $A = \{x: x \in \mathbb{R}, x \geq 1\}$.

  A function $f: A \rightarrow \mathbb{R}$ is defined as $f(x) = x^2 - x$. Show that f is one-to-one.

  (7 marks)

(b) Let $f(x) = 3x + 2$ and $g(x) = e^{2x}$.

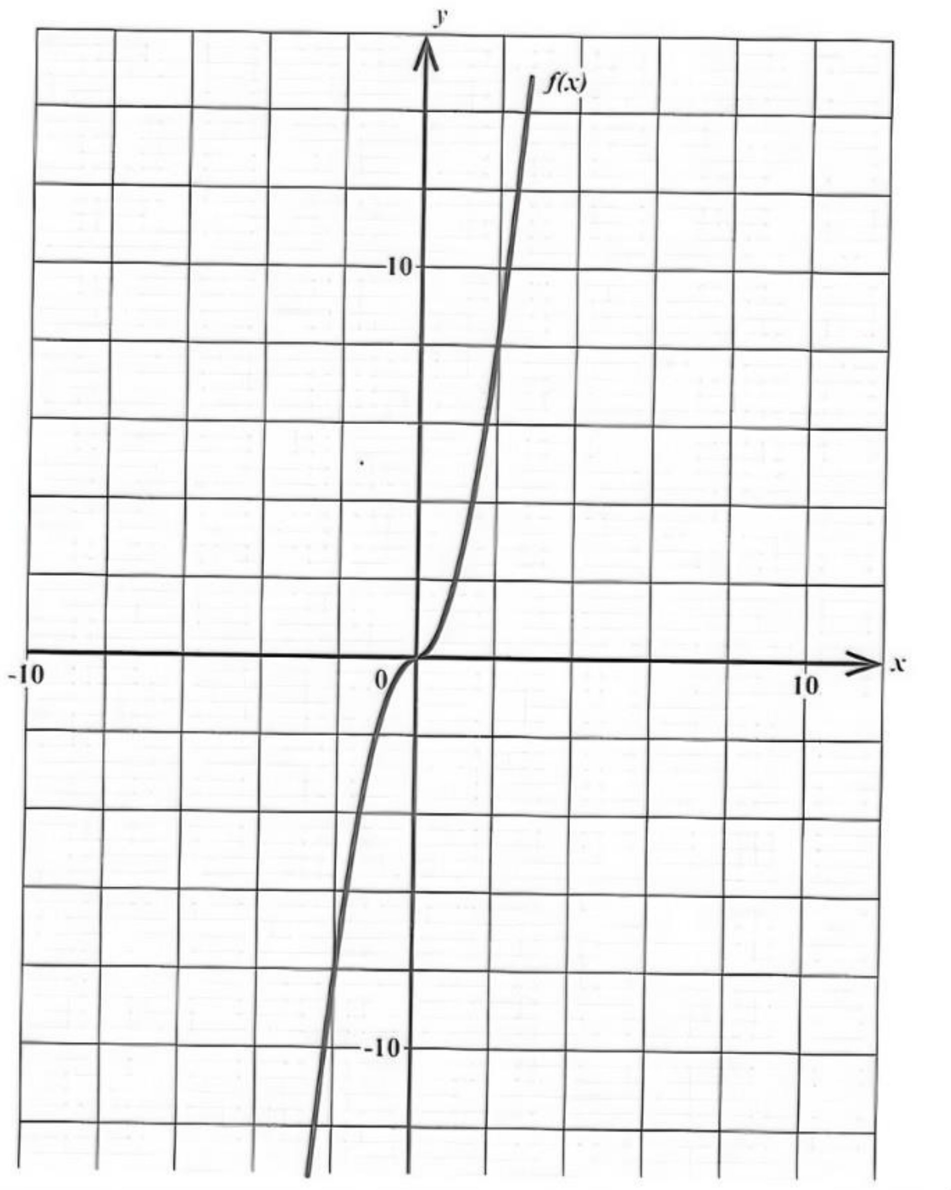
(i) Find

a) $f^{-1}(x)$ and $g^{-1}(x)$. (4 marks)

b) $f[g(x)]$ or $(f \cdot g(x))$. (1 mark)

- Past Paper Question – 2018 Paper 2 #2 a)

  The diagram below shows the graph of the function $f(x)$.



(a) On the diagram above,

(i) Sketch the inverse of $f(x)$.

(ii) Use a graphical method to show that f is bijective.

FUNCTIONS III

- Practice Question 2

  Solve $|5x + 1| = 3$

- Practice Question 3

  Solve $|3x - 1| = |x + 1|$

- Past Paper Question – 2013 Paper 2 #2 c) (ii)

  Solve $|x + 2| = 3x + 5$ (4 marks)

- Past Paper Question – 2005 Paper 2 #1

  Solve $x^2 - 2|x| - 3 = 0$

- Past Paper Question – 2001 Paper 2 #1

  Solve $|2x - 1| - 3 = x$

- Past Paper Question – 2003 Paper 2

  Solve $|2x - 3|^2 - 6|2x - 3| + 5 = 0$

- Past Paper Question – 2012 Paper 2 #1 c) (i)

  Solve, for real values of x, the inequality

  $$|3x - 7| \leq 5.$$ (5 marks)

  Hint: When an inequality is multiplied by a negative, the sign flips.

CUBIC EQUATIONS

- Past Paper Question – 2012 Paper 2 #2 b)

  The roots of the equation $4x^2 - 3x + 1 = 0$ are α and β.

  Without solving the equation

  (i) Write down the values of $\alpha + \beta$ and $\alpha\beta$. (2 marks)

  (ii) Find the value of $\alpha^2 + \beta^2$. (2 marks)

  (iii) Obtain a quadratic equation whose roots are $\frac{2}{\alpha^2}$ and $\frac{2}{\beta^2}$. (5 marks)

- Practice Question 1

  If $2x^3 - 5x^2 + 4x + 6 = 0$ has roots α, β and γ.

  Find:

  (i) $\alpha + \beta + \gamma =$

  (ii) $\alpha\beta + \alpha\gamma + \beta\gamma =$

  (iii) $\alpha\beta\gamma =$

  (iv) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} =$

- Practice Question 2

  If $x^3 + 4x - 3 = 0$ has roots α, β and γ.

  Find:

a) (i) $\alpha + \beta + \gamma$ (ii) $\alpha\beta + \alpha\gamma + \beta\gamma$ (iii) $\alpha\beta\gamma$

b) $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$

c) $\frac{1}{\alpha^2\beta^2} + \frac{1}{\alpha^2\gamma^2} + \frac{1}{\beta^2\gamma^2}$

d) $\frac{1}{\alpha^2\beta^2\gamma^2}$

e) An equation whose roots are $\frac{1}{\alpha^2}$, $\frac{1}{\beta^2}$ and $\frac{1}{\gamma^2}$

- Practice Question 3

  Given $x^3 + 5x + 2 = 0$ has roots α, β and γ. Find an equation whose roots are $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$.

- Past Paper Question – 2022 Paper 2 #2 d)

  The roots of the equation $2x^3 - x^2 + 3x - 1 = 0$ are α, β and γ.

  Given that $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{3}{2}$ and $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma} = \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}$, determine the equation whose roots are $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$. (6 marks)

- Past Paper Question – 2022 Paper 2 #2 c)

  The function $f(x) = 2x^3 - px^2 + qx - 10$ is divisible by $x - 1$ and has a remainder of -6 when divided by $x + 1$.

  Calculate the values of p and q. (6 marks)

- Past Paper Question – 2022 Paper 2 #2 b)

  Solve the equation $6 - \frac{7}{2^{2x}} - \frac{3}{4^{2x}} = 0$. (8 marks)

- Past Paper Question – 2022 Paper 2 #2 a)

  Let $f(x) = \frac{3x+1}{x}$ and $g(x) = e^{-2x} + 1$.

  (i) Show that $f^{-1}(x) = \frac{1}{x-3}$. (3 marks)

  (ii) Hence, or otherwise, write an expression for $f^{-1}[g(x)]$. (2 marks)

- Past Paper Question – 2022 Paper 2 #1 c)

  Solve the inequality $|3x + 2| \geq 4$. (5 marks)

- Past Paper Question – 2022 Paper 2 #1 d)

  Solve the logarithmic equation $\log_5(x + 2) + \log_5(x + 6) = 1$. (6 marks)

Module 2:
Trigonometry and Plane Geometry

TRIGONOMETRY I

- Practice Question 1

  Working in radians, solve $\sin\theta = \frac{1}{2}$ to obtain all solutions between the range $[0\,,2\pi]$.

- Practice Question 2

  (i) Sketch the graph for $y = 5\sin\frac{1}{2}\,x - 2$ such that $0 \leq x \leq 2\pi$.

(ii) Solve $5\sin\frac{1}{2}\,x - 2 = 0$ to obtain all solutions such that $0 \leq x \leq 2\pi$.

- Practice Question 3

  Without using identities, solve $4 - 4\sin^2\theta - 4\sin\theta - 1 = 0$ for $0 \leq \theta \leq \pi$.

- Past Paper Question – 2016 Paper 2 #3 a)

  (i) Show that $\sec^2\theta = \frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta - \sin\theta}$. (4 marks)

(ii) Hence, or otherwise, solve the equation $\frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta - \sin\theta} = \frac{4}{3}$

for $0 \leq \theta \leq 2\pi$. (5 marks)

- Past Paper Question – 2014 Paper 2 #3 a) (i)

  Prove that $\frac{\cot y - \cot x}{\cot x + \cot y} = \frac{\sin(x-y)}{\sin(x+y)}$. (4 marks)

TRIGONOMETRY II

- Practice Question 1

  Find the general solution for $\sin\theta = \frac{1}{2}$.

- Practice Question 2

  Find the general solution for $\sin 2\theta = \frac{\sqrt{3}}{2}$.

- Practice Question 3

  Find the general solution for $\tan 3\theta = 1$.

- Practice Question 4

  Find the general solution for $\cos(\theta - \frac{\pi}{3}) = \frac{1}{2}$.

- Practice Question 5

Find the general solution for $2\sin\theta + 5\cos\theta = \frac{\sqrt{29}}{2}$.

(i) Express in the form $R\sin(\theta + \alpha)$.

(ii) Use the general solution to solve $R\sin(\theta + \alpha) = \frac{\sqrt{29}}{2}$.

- Past Paper Question – 2021 Paper 2 #3 a)

  (i) Express $4\sin\theta + 3\cos\theta$ in the form of $R\sin(\theta + \alpha)$. (5 marks)

  (ii) Hence, solve the equation $4\sin\theta + 3\cos\theta = 2$. (7 marks)

- Practice Question 6

  Solve $3\cos\theta - 4\sin\theta = 3$.

TRIGONOMETRY III

- Past Paper Question – 2013 Paper 2 #3 b)

(i) Express $f(\theta) = 3\cos\theta - 4\sin\theta$ in the form $r\cos(\theta + \alpha)$ where

$r > 0$ and $0° \leq \alpha \leq \frac{\pi}{2}$. (4 marks)

(ii) Hence, find

a) The maximum value of $f(\theta)$. (2 marks)

b) The minimum value of $\frac{1}{8 + f(\theta)}$. (2 marks)

- Past Paper Question – 2016 Paper 2 #3 b)
  - (i) Express the function $f(\theta) = \sin\theta + \cos\theta$ in the form $r\sin(\theta + \alpha)$, where;

    $$r > 0 \text{ and } 0 \leq \theta \leq \frac{\pi}{2}.$$ (5 marks)

  - (ii) Hence, find the maximum value of f and the **smallest** non-negative value of θ at which it occurs. (5 marks)

- Practice Question 1

  Express $\sin 8x \cos 4x$ as a sum of trigonometric terms.

- Practice Question 2

  Prove that $\dfrac{\cos 3x + \cos x}{\sin 3x - \sin x} = \cot x$.

- Practice Question 3

  Prove that $\dfrac{\sin 4x - \sin 2x}{\cos 4x + \cos 2x} = \tan x$.

- Past Paper Question – 2022 Paper 2 #3 a)

  Prove that $\frac{\tan\theta \sin\theta}{1-\cos\theta} = 1 + \frac{1}{\cos\theta}$. (6 marks)

TRIGONOMETRY IV

- Past Paper Question – 2019 Paper 2 #3 a) b) c)

  a) Solve the equation $5\sec\theta - 2\sec^2\theta = \tan^2\theta - 1$ for $-\pi \le \theta \le \pi$.

  (7 marks)

b) Let $f(x) = \cos x + 2\sin x$.

(i) Express f in the form $R\sin(x + \alpha)$, where α is an acute angle.

(5 marks)

(ii) Hence, or otherwise, find the general solution of $\cos x + 2\sin x = 0$.

(3 marks)

(iii) Determine the **minimum** value of $\frac{2}{2-f}$. (3 marks)

c) Prove the identity $\tan(A + B) - \tan A = \dfrac{\sin B}{\cos A \cos(A+B)}$.

- Past Paper Question – 2022 Paper 2 #3 b) c)

b) Solve the equation $\tan^2 \theta - 2 \tan \theta = 3$ for $0 \leq \theta \leq 2\pi$. (6 marks)

c) (i) Show that $4\cos\theta + 3\sin\theta = 5\sin(\theta + 0.927°)$. (5 marks)

(ii) Hence, or otherwise, solve the equation $4\cos\theta + 3\sin\theta = 0$.
(4 marks)

- Past Paper Question – 2017 Paper 2 #3 a) b) c)

a) (i) Prove the identity $\tan(A+B) \equiv \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$. (4 marks)

(ii) Given that $\sin A = \frac{3}{5}$ and $\cos B = \frac{-1}{2}$ where angle A is acute and angle B is obtuse, express $\tan(A + B)$ in the form $a + b\sqrt{3}$, where a and b are real numbers. (6 marks)

b) Solve the equation $\sin^2\theta - 2\cos^2\theta + 3\cos\theta + 5 = 0$ for $0 \leq \theta \leq 4\pi$.

(6 marks)

c) (i) Express $f(\theta) = 6\cos\theta + 8\sin\theta$ in the form $r\sin(\theta + \alpha)$ where $0 \leq \alpha \leq 90°$. (3 marks)

(ii) Hence, or otherwise, find the general solution of $f(\theta) = 2$. (6 marks)

CO-ORDINATE GEOMETRY I

- Practice Question 1

  Given that the center, c of a circle is $(2, -3)$, find the equation of the tangent at the point $(3, 1)$.

- Practice Question 2

  Find the equation of the circle given

  $P = (2, 3) \quad , \; Q = (5, 6) \quad , \; R = (8, 3)$

- Practice Question 3

  Given that the Parametric equation of a circle is given by:

$$x = 5 + 2\cos\theta$$

$$y = 3 + 2\sin\theta$$

  Obtain the Cartesian equation using the following steps:

  - Step 1: Find $\cos\theta$
  - Step 2: Find $\sin\theta$
  - Step 3: Substitute steps 1 and 2 into $(\cos^2\theta + \sin^2\theta = 1)$

CO-ORDINATE GEOMETRY II

- Practice Question 1

  Sketch the following ellipse

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$$

- Practice Question 2

  Sketch the following ellipse

$$\frac{(x+2)^2}{25} + \frac{(y-3)^2}{16} = 1$$

- Practice Question 3

  Given that the Parametric equation of an ellipse is given by:

$$x = 4 + 2\cos\theta$$

$$y = 3 + 4\sin\theta$$

(i) Obtain the Cartesian equation.

Hint: Follow the steps used to obtain the Cartesian equation of a circle.

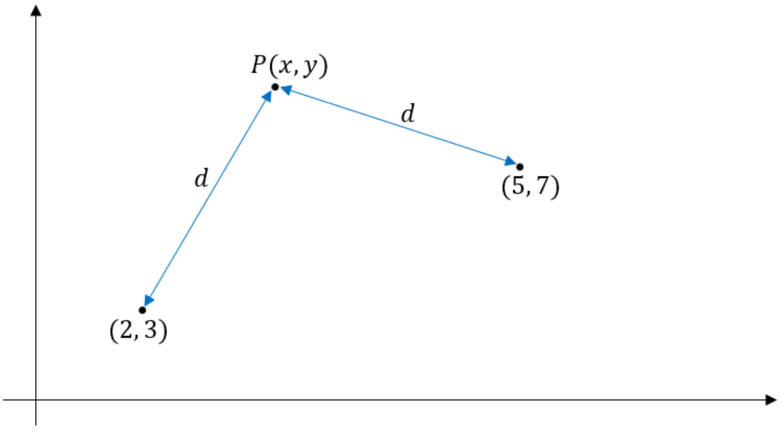
(ii) Define the ellipse.

Hint: State its center and foci.

CO-ORDINATE GEOMETRY III

- Practice Question 1

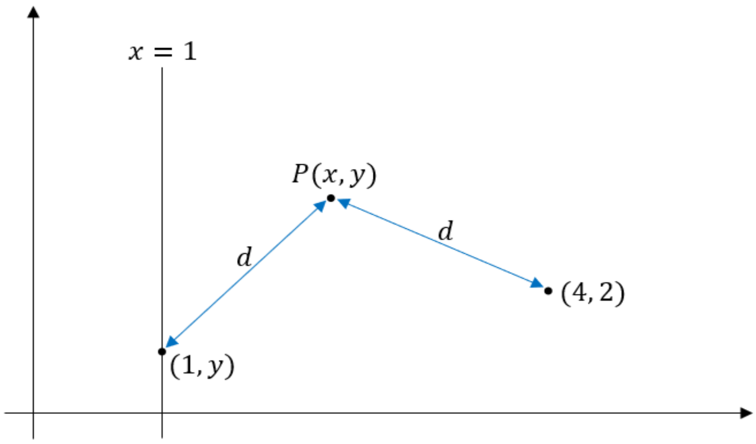
  A point $P(x, y)$ is always the same distance, d from the points $(2, 3)$ and $(5, 7)$ as seen on the following diagram. *(Not drawn to scale)*



  Determine and describe the equation of the locus at the point $P(x, y)$.

- Practice Question 2

A point $P(x, y)$ is always equidistant from the line $x = 1$ and the point $(4, 2)$ as seen on the following diagram. *(Not drawn to scale)*



Determine and describe the equation of the locus at the point $P(x, y)$.

- Past Paper Question – 2015 Paper 2 ROR #4 b)

  A point $P(x, y)$ moves so that its distance from the fixed point $(0, 3)$ is two times the distance from the fixed point $(5, 2)$. Show that the equation of the locus of the point $P(x, y)$ is a circle. (12 marks)

- Past Paper Question – 2016 Paper 2 #4 c)

  A point, $P(x, y)$, moves such that its distance from the x-axis is half its distance from the origin.

  Determine the Cartesian equation of the locus of P. (5 marks)

- Past Paper Question – 2015 Paper 2 ROR #4 a)

  The circles C_1 and C_2 are defined by the parametric equations as follows:

  $$C_1: \quad x = \sqrt{10}\cos\theta - 3; \qquad y = \sqrt{10}\sin\theta + 2$$

  $$C_2: \quad x = 4\cos\theta + 3; \qquad y = 4\sin\theta + 2$$

(i) Determine the Cartesian equations of C_1 and C_2 in the form $(x-a)^2 + (y-b)^2 = r^2$. (4 marks)

(ii) Hence, or otherwise, find the points of intersection of C_1 and C_2. (9 marks)

VECTORS I

- Practice Question 1

  If $\bar{a} = 3i - k$ and $\bar{b} = 2i - 6j$, find $\bar{c}$ if $\bar{a} + \bar{b} = \bar{c}$.

- Practice Question 2

  Let $u = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ and $v = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$. Find the angle between u and v.

- Past Paper Question – 2014 Paper 2 #4 c)

  The points $P\ (3, -2, 1)$, $Q\ (-1, \lambda, 5)$ and $R\ (2, 1, -4)$ are three vertices of a triangle PQR.

  (i) Express EACH of the vectors $\overrightarrow{PQ}$, $\overrightarrow{QR}$ and $\overrightarrow{RP}$ in the form $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.

  (4 marks)

  (ii) Hence, find the value of λ, given that PQR is right-angled with the side PQ as hypotenuse. (6 marks)

- Practice Question 3

  Given that a line, L, passes through the point $A\ (2, 1, -3)$ and is parallel to the vector $2i + j - k$, find for the line, L:

  (i) Vector Equation

  (ii) Parametric Equation

  (iii) Cartesian Equation

VECTORS II

- Practice Question 1

  Given that the Cartesian equation of the line, L is

$$\frac{x + 1}{3} = \frac{y - 2}{-5} = \frac{z - 1}{4}$$

  Obtain the following:

  (i) Parametric Equation

  (ii) Vector Equation

- Practice Question 2

The vector equation of two lines, L_1 and L_2, are:

L_1: $r = -5\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k} + \lambda(6\boldsymbol{i} - 3\boldsymbol{j} + 9\boldsymbol{k})$

L_2: $r = -5\boldsymbol{i} - 4\boldsymbol{j} + 3\boldsymbol{k} + \mu(-2\boldsymbol{i} + 3\boldsymbol{j} - 5\boldsymbol{k})$

The above vector equations have been converted to Parametric form for you.

$$L_1:\ r = \begin{pmatrix} -5 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -3 \\ 9 \end{pmatrix} \quad \therefore \begin{cases} x = -5 + 6\lambda \\ y = \ \ 2 - 3\lambda \\ z = -3 + 9\lambda \end{cases}$$

$$L_2:\ r = \begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ -5 \end{pmatrix} \quad \therefore \begin{cases} x = -5 - 2\mu \\ y = -4 + 3\mu \\ z = \ \ 3 - 5\mu \end{cases}$$

Determine the point of intersection of the lines L_1 and L_2.

- Past Paper Question – 2017 Paper 2 #4 c)

  The vector equation of two lines, L_1 and L_2, are:

  $L_1 = -\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k} + \alpha(-2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k})$

  $L_2 = -2\boldsymbol{i} + \boldsymbol{j} - 4\boldsymbol{k} + \beta(\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$

  (i) Show that L_1 and L_2 intersect. (5 marks)

  (ii) Hence, determine the coordinates of the point of intersection of the two lines. (2 marks)

- Past Paper Question – 2018 Paper 2 #4 c)

  (i) Determine the vector equation of a plane which passes through the point $(1, 3, 0)$ and which is perpendicular to the vector $2i + 4j + 5k$.

  (3 marks)

  (ii) Obtain the Cartesian equation of the vector equation found in part (i).

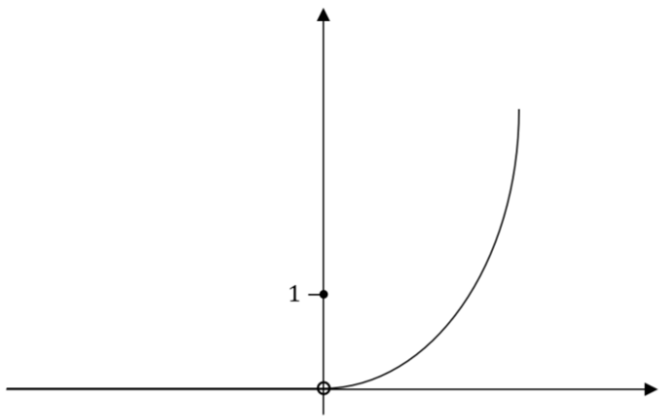
Module 3:
Calculus I

LIMITS I

- Practice Question 1

A function is given as $f(x) = \begin{cases} 0, & x < 0 \\ 1, & x = 0 \\ x^2, & x > 0 \end{cases}$

The graph sketch of the above piecewise function is as follows. *(Not drawn to scale)*



(i) Does the limit exist at $x = 0$?

$\lim_{x \to 0^+} f(x) =$ ______ $\lim_{x \to 0^-} f(x) =$ ______

(ii) Is the function continuous at $x = 0$?

- Practice Question 2

A function is given as $f(x) = \begin{cases} 0, & x < -1 \\ x^2 - 1, & -1 < x < 2 \\ 5 - x, & x \geq 2 \end{cases}$

(i) Sketch the graph of the above piecewise function.

(ii) State if $f(x)$ is continuous at:

a) $x = -1$

b) $x = 2$

- Practice Question 3

  The function f on $\boldsymbol{R}$ is defined by:

$$f(x) = \begin{cases} x^2 + 1 & if\ x \geq 2 \\ bx - 1 & if\ x < 2 \end{cases}$$

  Determine the value of b such that $f(x)$ is continuous at $x = 2$.

LIMITS II

- Practice Question 1

$$\lim_{x \to \infty} \frac{3x^2 + 1}{2x - 1}$$

- Practice Question 2

$$\lim_{x \to 2} \frac{x^2 - 4}{2x^2 - 4x}$$

- Practice Question 3

Given that $\lim_{x \to 3} [\, f(x) + 2x] = 8$, find $\lim_{x \to 3} [4 \cdot f(x)]$

- Practice Question 4

$$\lim_{x \to 0} \frac{\sin 4x}{\sin 5x}$$

- Past Paper Question – 2012 Paper 2 #5 a) (iii)

  By using the fact that $\lim_{x \to 0} \frac{\sin x}{x} = 1$, or otherwise, find

$$\lim_{x \to 0} \frac{2x^3 + 4x}{\sin 2x}$$

(5 marks)

- Past Paper Question – 2015 Paper 2 ROR #5 a)

  Let f be a function defined as

$$f(x) = \begin{cases} \dfrac{\sin(ax)}{x} & if\ x \neq 0, \quad a \neq 0 \\ 4 & if\ x = 0 \end{cases}$$

  If f is continuous at $x = 0$, determine the value of a. (4 marks)

- Past Paper Question – 2018 Paper 2 #6 a)

  A function f is defined as

$$f(x) = \begin{cases} \frac{x^4-1}{x-1}, & x < 1 \\ 4x, & x > 1 \\ 2, & x = 1 \end{cases}$$

(i) Determine whether or not the limit of f at $x = 1$ exists. (4 marks)

(ii) Determine whether f is continuous at $x = 1$. (2 marks)

DIFFERENTIATION I

- Past Paper Question – 2016 Paper 2 #6 c)

  Use first principles to differentiate $f(x) = \sqrt{x}$ with respect to x. (6 marks)

- Practice Question 1

  Use first principles to differentiate $f(x) = \sin x$.

- Practice Question 2

  Use first principles to differentiate $f(x) = \cos 2x$.

DIFFERENTIATION II

- Practice Question 1

  The equation of a curve is given as $f(x) = x^4 - 2x^3 + 2$.

  Using the second derivative test first, calculate the coordinates of the stationary points of $f(x)$ and determine the nature of these stationary points.

- Practice Question 2

$$\frac{d(\ln|3x|\,)}{dx}$$

- Practice Question 3

$$\frac{d(\ln\,|3x^2+1|\,)}{dx}$$

- Practice Question 4

$$\frac{d(\ln|\cos x|\,)}{dx}$$

- Practice Question 5

  A funnel filled with water is in the shape of an inverted circular cone of base radius $8cm$ and height $15cm$. Water is leaking away through a small hole in the vertex at a constant rate of $2cm^3/s$. Let Vcm^3 be the volume of water in the funnel and $h\ cm$ be the depth of the water in the funnel.

  a) Express V in terms of h.

  b) Find the rate of change of the depth of the water in the funnel when the depth of the water is $10cm$.

INTEGRATION

- Practice Question 1

  Use integration by substitution to determine $\int (4x - 1)^3 \cdot dx$.

- Practice Question 2

  Use integration by substitution to determine $\int x \cdot (2x^2 - 1)^3 \cdot dx$.

- Past Paper Question – 2018 Paper 2 #5 a)

  Use the substitution $u = x^4 + 2$ to determine $\int (x^4 + 2)^3 \cdot (4x)^7 \cdot dx$.

  (6 marks)

- Practice Question 3

  Solve $\frac{dy}{dx} = \frac{\sin x}{\cos y}$, $\quad y\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$

- Past Paper Question – 2018 Paper 2 #6 c)

  The slope of a tangent of the curve, g, at any point (x, y) is equal to the product of the x-coordinate and the reciprocal of the y-coordinate of the point of tangency.

  (i) Formulate an appropriate differential equation and find the equation of the curve family. (5 marks)

  (ii) Hence, or otherwise, determine the equation of the curve given that it passes through the point $(1, 3)$. (3 marks)