Additional Mathematics Summary

Section 1: Algebra, Sequences and Series

A. Algebra
B. Quadratics
C. Inequalities
D. Surds, Indices and Logarithms
E. Series and Sequences

Section 2: Coordinate Geometry, Vectors and Trigonometry

A. Coordinate Geometry
B. Vectors
C. Trigonometry

Section 3: Introductory Calculus

A. Differentiation
B. Integration

Section 4: Probability and Statistics

A. Data representation and Analysis
B. Probability Theory

Section 1: Algebra, Sequences and Series

A. Algebra

Remainder theorem: If f(x) is divided by (ax+b) then the remainder is f $(\frac{-b}{a})$.
Factor Theorem: If f(x) is divided by (ax+b) and the remainder is 0, then (ax+b) is a factor.

long division:

Theorem:

To solve or factorise a cubic:

a) Use trial and error to obtain first factor.
2. Divide function by first factor.
3. Solve or factorize the quadratic obtained.

B. Quadratics

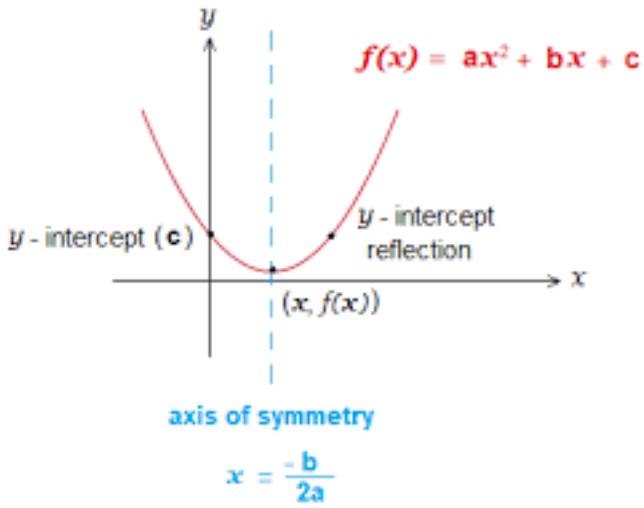
General form: $y = ax^2 + bx + c$ [highest power of x is 2]

To complete the square: $y = a(x + h)^2 + k$. where $h = \frac{b}{2a}$ and $k = c - ah^2$

To sketch a quadratic:

- Shape: minimum , $a > 0$
  Maximum, $a < 0$

- Turning point : (- h , k)
- Maximum or minimum value is always k .
- X-value which gives max or minimum value is – h.
- X-intercepts: solve $ax^2 + bx + c = 0$
- Y-intercept: (0,c)



To determine the **nature of the roots** use the discriminant, $D = b^2 - 4ac$:

- If $b^2 - 4ac > o$ then you have 2 real and distinct roots.
- If $b^2 - 4ac = 0$ then you have real and equal roots, i.e. one root.
- If $b^2 - 4ac < 0$ then you no real roots.

Some equations can be **reduced to a quadratic** then solved by using an appropriate substitution.
e.g. $x^4 - 2x^2 + 5 = 0$ can be reduced to $p^2 - 2p + 5 = 0$, if we let $p = x^2$ and $p^2 = x^4$.

If $\boldsymbol{\alpha}$ ***and*** $\boldsymbol{\beta}$ are the roots of $ax^2 + bx + c = 0$ then :

$\alpha + \beta = \frac{-b}{a}$ $\alpha\beta = \frac{c}{a}$ $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

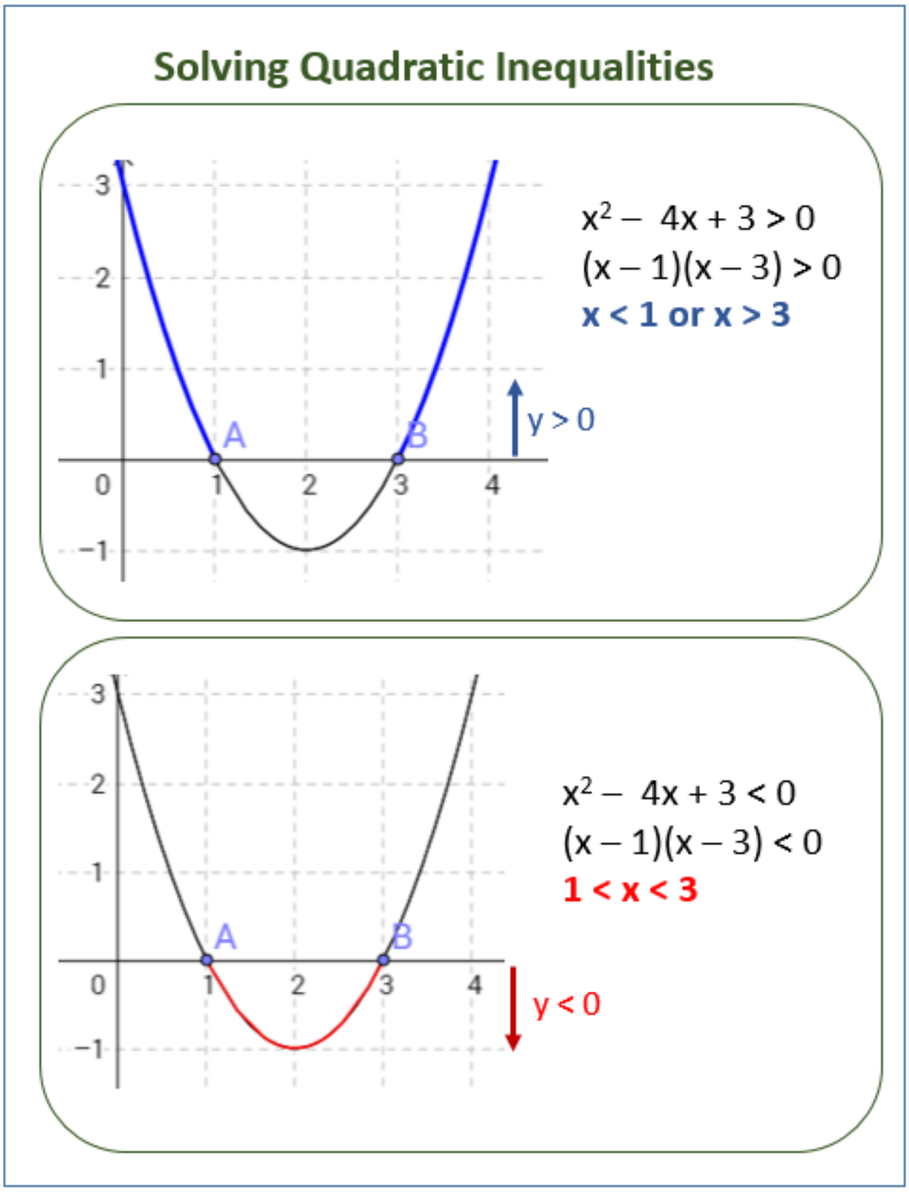
$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$

C. Inequalities

To solve quadratic inequalities:

1. Solve quadratic to obtain critical values, α *and* β.
2. Solution for f(x) > 0: $x < \alpha \;\cup\; x > \beta$
3. Solution for f(x) < 0: $\alpha < x < \beta$

NB. For the above rules to work the quadratic must be a minimum.



For rational inequalities of the form $\frac{ax+b}{cx+d} > k$, multiply both sides by denominator squared, then simplify and solve.

$$\frac{ax+b}{cx+d} \times (cx+d)^2 > k \times (cx+d)^2$$

D. Surds, Indices and Logarithms

indices

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $a^0 = 1$
- $(a^m)^n = a^{mn}$
- $\frac{1}{a^m} = a^{-m}$
- $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

surds

- $\sqrt{a}\times\sqrt{b} = \sqrt{ab}$
- $\sqrt{a}\times\sqrt{a} = a$
- $\frac{a}{\sqrt{a}} = \sqrt{a}$
- rationalise:
- $\frac{a+\sqrt{b}}{c+\sqrt{d}} \times \frac{c-\sqrt{d}}{c-\sqrt{d}}$
- $(c+\sqrt{d})(c-\sqrt{d})$
  $= c^2 - (\sqrt{d})^2$
  $= c^2 - d$

logs

- exp. to logs.
- $a^c = b \leftrightarrow \log_a b = c$
- $\log_a a = 1$
- $\log_a 1 = 0$
- $\log_a m^n = n\log_a m$
- $\log_a m + \log_a n = \log_a mn$
- $\log_a m - \log_a n = \log_a \frac{m}{n}$

Linear reduction: logarithms can be used to linearize equations, ie convert to form y = mx+c.

Example: 1. $y = ab^x$ can be transformed to $\lg y = x \lg b + \lg a$:
$Y = \lg y$, $X = x$, $m = \lg b$, $c = \lg a$

2. $y = ax^b$ can be transformed to $\lg y = b \lg x + \lg a$:
$Y = \lg y$, $X = \lg x$, $m = b$, $c = \lg a$

NB. - Y and X must never contain the unknown constants

E. Series and Sequences

Summation notation: $\sum_{r=1}^{5}(2r+1) = (2(1)+1\,) + (2(2)+1) + (2(3)+1) + (2(4)+1)$
$= 3 + 5 + 7 + 9$
$= 24$

Rules of Summation:

- $\sum k a_n = k \sum a_n$
- $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n$

	AP	**GP**
Parameters	a – first term d – common difference	a – first term r – common ratio
Sequence	a, a + d , a + 2d, a + 3d,……	a, ar, ar^2, ar^3, ar^4,……..
General Term	$T_n = a + (n-1)d$	$T_n = ar^{n-1}$
Sum of the first n terms	$S_n = \frac{n}{2}\,[2a + (n-1)d]$ $S_n = \frac{n}{2}\,[a + l\,]$ l – last term	$S_n = \frac{a(1-r^n)}{1-r}$, r < 1 $S_n = \frac{a(r^n-1)}{r-1}$, r > 1
Convergent or Divergent	Divergent	Convergent iff : -1 < r < 1
Sum to Infinity	Does not exist	$S_\infty = \frac{a}{1-r}$

Section 2: Coordinate Geometry, Vectors and Trigonometry
A. Coordinate Geometry

Distance between two points: $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Mid-point: $\left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$

Gradient: m = $\frac{y_2-y_1}{x_2-x_1}$ $\frac{\text{parallel lines have equal gradients } [m_1 = m_2]}{\text{perpendicular lines, product of gradients equal } -1. \ [m_1 m_2 = -1]}$

Equation of a line: y = mx + c m – gradient
c – y-intercept (cuts the y-axis)

To find the equation of a line:
- find gradient of line
- obtain a point on the line
- substitute in $y - y_1 = m(x - x_1)$

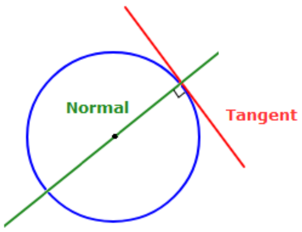
NB.
- *Solving equations simultaneously gives the points of intersection of the equations.*
- *If a line is a tangent to a curve, then when solved simultaneously, only 1 solution is obtained.*

Equation of a circle:

General form: $(x + f)^2 + (y + g)^2 = r^2$ centre = (-f , -g) ; radius = r

Or

$x^2 + y^2 + 2fx + 2gy + c = 0$ centre = (-f , - g) ; radius = $\sqrt{f^2 + g^2 - c}$



To determine the equation of tangent at a point P (x, y) on a circle:
- Find the gradient of the radius using the centre and point P
- Find the gradient of the tangent using $\frac{-1}{\text{gradient of radius}}$
- Substitute gradient and point P in $y - y_1 = m(x - x_1)$

NB – To determine the equation of the normal at a point P (x, y) on a circle, use the same steps as above, except that gradient of radius = gradient of normal.

B. Vectors

General form: For a point P (x, y), its position vector can be written as: $\overline{OP} = \binom{x}{y}$ = xi + yj

$\overline{P0} = \binom{-x}{-y}$

Equal vectors: If two vectors are equal:

ai + bj = ci + dj
then a = c and b = d.

To add or subtract vectors: $\binom{a}{b} \pm \binom{c}{d} = \binom{a \pm c}{b \pm d}$

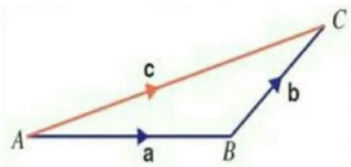
Multiply by a scalar: $k\binom{a}{b} = \binom{ka}{kb}$

Multiplying two vectors: $\binom{a}{b}\binom{c}{d} = ac + bd$ This is also known as scalar product.

Magnitude of a vector: If we have a vector xi + yj then its magnitude is given by $\sqrt{x^2 + y^2}$

Unit vector: A unit vector has a magnitude of one and is obtained by: $\frac{\bar{A}}{\|A\|}$

Displacement Vectors:



In the vector diagram above, the triangle can be used to determine unknown vectors by finding an alternative path along the triangle.

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

$$\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$$

$$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$$

Angle Between Two Vectors: The angle between two vectors $\bar{a}$ and $\bar{b}$ is given by

$\cos\theta = \frac{a.b}{\|a\|\|b\|}$

If two vectors $\bar{a}$ and $\bar{b}$ are parallel, then they are scalar multiples of each other.
If two vectors $\bar{a}$ and $\bar{b}$ are perpendicular, then a.b = 0.

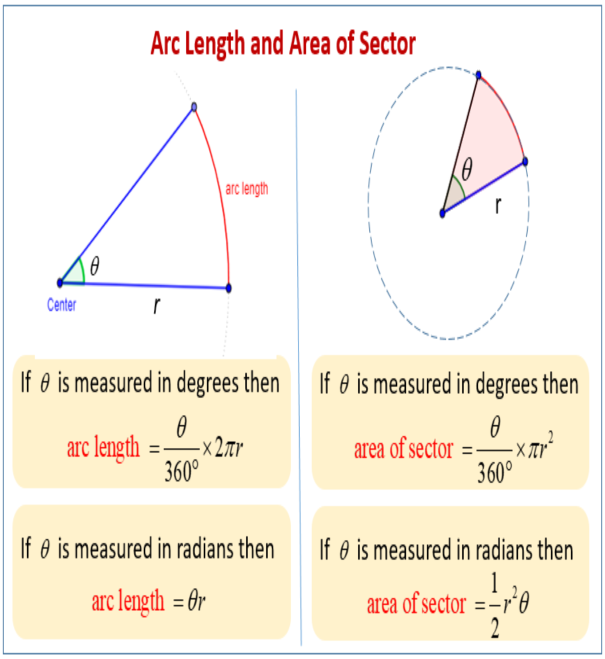
C. Trigonometry

Radians, Arcs, and sectors

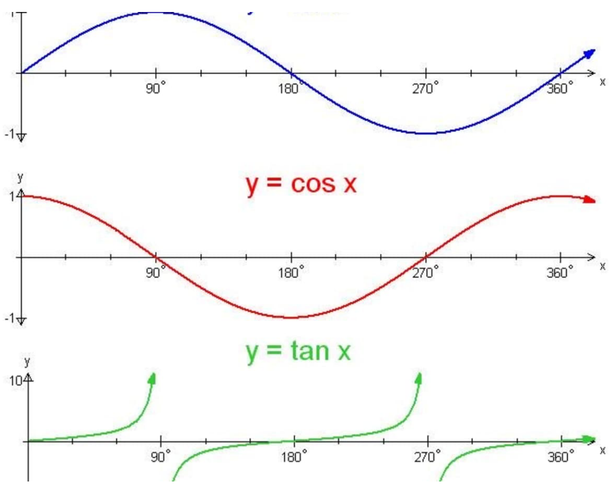
Radians/Degrees Conversions

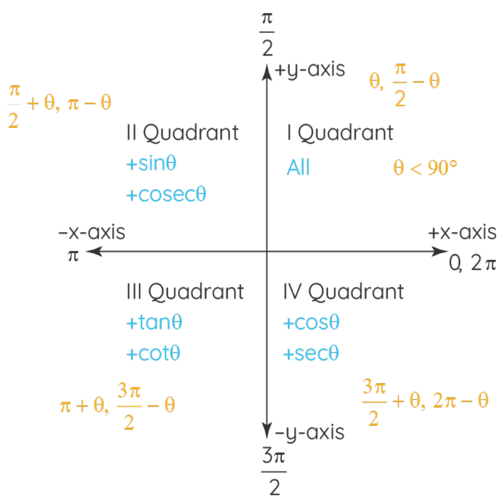
These are worth actually LEARNING

Angle in degrees	Angle in radians
180	π
30	$\frac{\pi}{6}$
45	$\frac{\pi}{4}$
60	$\frac{\pi}{3}$
90	$\frac{\pi}{2}$
360	2π
Approximately 57.3	1



Trig. Graphs





Identities

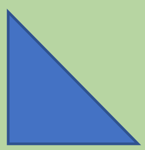
$\text{Tan } \theta = \frac{Sin\ \theta}{Cos\ \theta}$

$\sin^2\theta + \cos^2\theta = 1$
$\sin^2\theta = 1 - \cos^2\theta$
$\cos^2\theta = 1 - \sin^2\theta$

$\text{Sin } \theta = \frac{opp}{hyp}$

$\text{Cos } \theta = \frac{adj}{hyp}$

$\text{Tan } \theta = \frac{opp}{adj}$



More Identities

$\text{Sin } (A \pm B) = \text{Sin A Cos B} \pm \text{Sin B Cos A}$

$\text{Cos } (A \pm B) = \text{Cos A Cos B} \mp \text{Sin A Sin B}$

$\text{Tan } (A \pm B) = \frac{Tan\ A \pm Tan\ B}{1 \mp Tan\ A\ Tan\ B}$

Even More Identities

$\text{Sin 2A} = 2 \text{ Sin A cos A}$

$\text{Cos 2A} = \text{Cos}^2 A - \text{Sin}^2 A$
$= 1 - 2\text{Sin}^2 A$
$= 2\text{Cos}^2 A - 1$

$\text{Tan 2A} = \frac{2\ Tan\ A}{1 - Tan^2 A}$

NB

- When proving identities, it is sometimes helpful to expand and simplify both sides until they are equivalent
- When proving identities with cos 2A, remember to choose the identity based on if you need cosA *only* or sinA *only*.
- When solving trig. functions, pay attention to range in which the unknown angle lies.
- Learn these exact values:

$\cos(0) = \sin(90) = 1$ $\qquad$ $\cos(30) = \sin(60) = \frac{\sqrt{3}}{2}$

$\cos(60) = \sin(30) = \frac{1}{2}$ $\qquad$ $\cos(45) = \sin(45) = \frac{\sqrt{2}}{2}$

$\tan(30) = \frac{1}{\sqrt{3}}$ $\qquad$ $\tan(45) = 1$

$\tan(60) = \sqrt{3}$

Section 3: Introductory Calculus
A. Differentiation

How to differentiate?

- $y = ax^n$

  $\frac{dy}{dx} = nax^{n-1}$

  1. Bring down the power
  2. Write back the a
  3. Minus 1 from the power

- $y = ax$

  $\frac{dy}{dx} = a$

- $y = a$

  $\frac{dy}{dx} = 0$

- $y = (ax + b)^n$

  1.Bring down the power
  2. differentiate inside the bracket
  3. Put back the term in bracket
  4. minus 1 from the power

  $\frac{dy}{dx} = n(a)(ax + b)^{n-1}$

- $y = \cos x$

  $\frac{dy}{dx} = -\sin x$

  $y = \cos(ax+b)$

  $\frac{dy}{dx} = -a\sin(ax+b)$

- $y = \sin x$

  $\frac{dy}{dx} = \cos x$

  $y = \sin(ax+b)$

  $\frac{dy}{dx} = a\cos(ax+b)$

Product rule:

$y = u.v$ then $\frac{dy}{dx} = u.\frac{dv}{dx} + v.\frac{du}{dx}$

Quotient rule:

$y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Chain Rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Second derivative: $\frac{d^2y}{dx^2} = \frac{d(\frac{dy}{dx})}{dx}$

Applications of differentiation

To find the gradient :

$$\frac{dy}{dx} = gradient\ function$$

At any point on a function substitute the x-valve into the gradient function to obtain the **gradient of the tangent at that point**.

Displacement- time graph

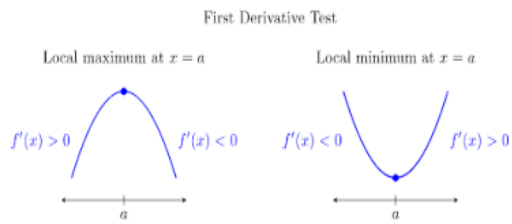
- gradient = velocity
- horizontal line = stationary
- negative gradient = change in direction

Velocity- time graph

- gradient = acceleration
- horizontal line = no acceleration
- area under graph = distance covered
- negative gradient= retardation

To determine the nature of stationary points:

When $\frac{dy}{dx} = 0$, then stationary point exists. If :



Also:

$$\frac{d^2y}{dx^2} > 0\ then\ point\ is\ minimum$$

And

$$\frac{d^2y}{dx^2} < 0\ then\ point\ is\ maximum$$

Rates of change:

Rate of change of x = $\frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dt}$

Rate of change Area, $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

Rate of change Volume, $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

Rate of change of y = $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

Kinematics:

Rate of change of displacement = velocity

Velocity, v = $\frac{ds}{dt}$

Rate of change of velocity = acceleration

Acceleration, a = $\frac{dv}{dt} = \frac{d^2s}{dt^2}$

B. Integration

How to integrate?

- $\int ax^n.dx = \frac{ax^{n+1}}{n+1}$ + c
  1. Add 1 to the power
  2. Divide by new power

- $\int a.dx$ = ax + c

- $\int (ax+b)^n.dx = \frac{(ax+b)^{n+1}}{a(n+1)}$
  1. Add 1 to the power
  2. Divide by the new power
  3. Divide by the differential of bracket

- $\int_a^b f(x).dx = {}_a^b[F(x)] = F[b] - F[a]$

- $\int \cos(ax+b).dx = \frac{1}{a}$ sin (ax + b) + c.
  1. $\frac{1}{differential\ of\ (ax+b)}$
  2. Integral of ***cos*** = **sin**
  3. Combine step 1 and 2 and put back (ax + b).

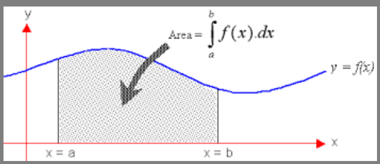
- $\int \sin(ax+b).dx = -\frac{1}{a}$ cos (ax + b) + c

Applications of Integration

1. To formulate the equation of a curve given its gradient function.

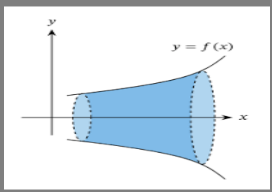
$$\int \frac{dy}{dx}.dx = y$$

2. Area under the curve:



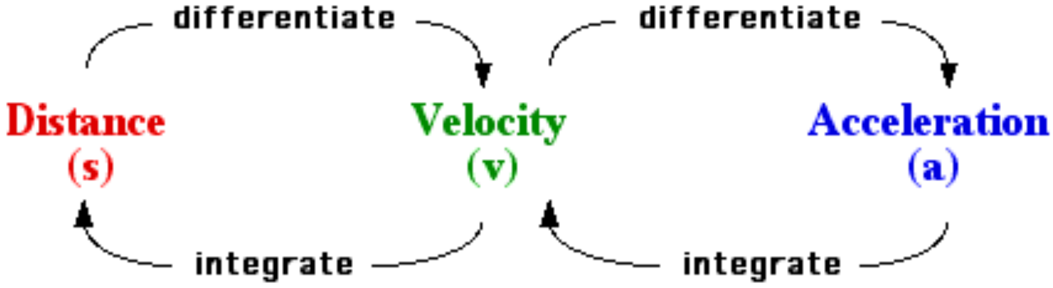
Area = $\int_a^b f(x)$

3. Volume of revolution about the x-axis:



Volume = $\pi \int_a^b y^2 .dx$

Kinematics



- Average velocity = $\frac{change\ in\ displacement}{time\ taken}$

- Average speed = $\frac{total\ distance\ travelled}{time\ taken}$

- A particle is instantaneously at **rest** when its **velocity is zero**.
- A particle reaches **maximum velocity** when its **acceleration is zero**.

Section 4: Probability and Statistics

A. Data representation and Analysis

Types of data:

Quantitative data: numerical data. e.g., Age
Qualitative data: non-numerical data, e.g. Gender.

Discrete data: takes the form of whole number values only.
Continuous data: can take any value within a particular range.

Quartiles:

Lower Quartile, $Q_1 = \frac{1}{4}\ (n+1)^{th}\ term$

Median, $Q_2 = \frac{1}{2}\ (n+1)^{th}\ term$

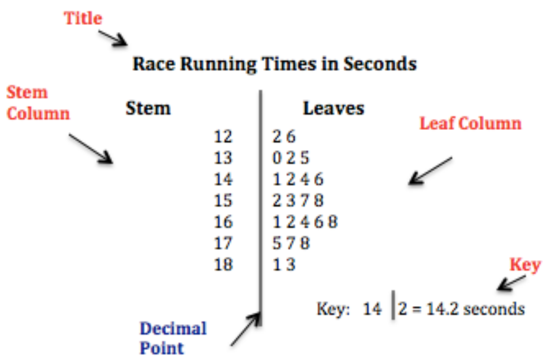
Upper Quartile, $Q_3 = \frac{3}{4}\ (n+1)^{th}\ term$

Interquartile Range = $Q_3 - Q_1$

Semi-Interquartile Range = $\frac{Q_3 - Q_1}{2}$

Stem and Leaf diagram

-All intervals must have equal widths
- Must have a key
-All the raw data is available



For grouped data:

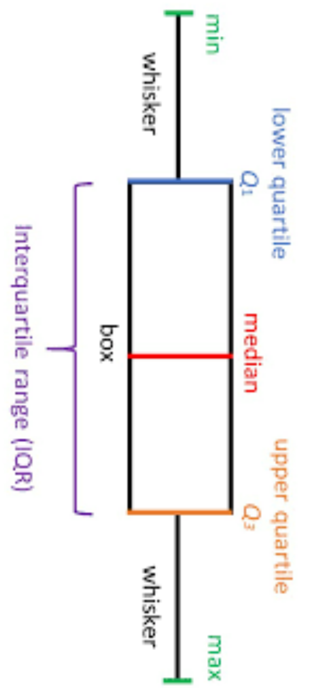
Mean, $\mu = \frac{\Sigma f(x)}{\Sigma f}$

Variance, $\sigma^2 = \frac{\Sigma f x^2}{\Sigma f} - \mu^2$

Standard deviation, $\sigma = \sqrt{variance}$

Box and whisker diagram

- Raw data is lost



B. Probability Theory

Classical Probability:

Probability of event A occurring = P (A) = $\frac{n\,(A)}{n\,(S)}$ $= \frac{no\ of\ outcomes\ in\ event\ A}{total\ number\ of\ possible\ outcomes}$

Basic Laws of Probability:

- Sum of the probabilities of all the outcomes in a sample space is equal to one. $\sum p = 1$.
- For any event A; $0 \leq P(A) \leq 1$.
- P(A) + P($\bar{A}$) = 1 , where P($\bar{A}$) is the probability of A not occurring

For Two Events:

- P(A or B) = P (A∪B) = P(A) + P(B) – P(A∩B)
- If P(A∩B) = 0 then events A and B are said to be mutually exclusive.
- For Conditional probability, A given B,

$$P(A/B) = \frac{P(A\cap B)}{P(B)}$$

- If A does not depend on B then A and B are said to be independent and

$$P(A\cap B) = P(A).P(B)$$

- Probability tree diagram :

  P(A∩ B) = P(A).P(B/A)

  P(B) = P(A).P(B/A) + P($\bar{A}$).P(B/$\bar{A}$)