A CONCISE COURSE

FOR CAPE

PURE MATHEMATICS

UNIT 1

NICHOLAS PAUL

<u>Section 1 – Reasoning and logic</u>

A proposition is a statement that makes a declaration that is either true or false, but not both.

A simple statement is a statement that cannot be decomposed into separate statements.

A compound statement is formed from two or more simple statements.

Algebra of propositions:

<u>Idempotent laws</u>

i) $p \vee p \equiv p$ **ii)** $p \wedge p \equiv p$

<u>Associative laws</u>

i) $(p \vee q) \vee r \equiv p \vee (p \vee r)$ **ii)** $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

<u>Commutative laws</u>

i) $p \vee q \equiv q \vee p$ **ii)** $p \wedge q \equiv q \wedge p$

<u>Distributive laws</u>

i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

ii) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Identity laws

i)$p \vee F \equiv p$ **ii)**$p \wedge T \equiv p$ **iii)**$p \vee T \equiv T$ **iv)**$p \wedge F \equiv F$

Complement laws

i)$\sim \sim p \equiv p$

ii)$\sim T \equiv F$, $\sim F \equiv T$

iii)$p \wedge \sim p \equiv T$

iv)$p \wedge \sim p \equiv F$

De Morgan's laws

i)$\sim(p \vee q) \equiv \sim p \wedge \sim q$

ii)$\sim(p \wedge q) \equiv \sim p \vee \sim q$

Absorption laws

i)$p \wedge (p \vee q) \equiv p$

ii)$p \vee (p \wedge q) \equiv p$

T or 1 represents true.

F or 0 represents false.

~q, q', $\bar{q}$, **negation** of q: 'It is not the case that q'.

∧, 'and' **conjunction**: p ∧ q is true only when both *p* and *q* are true.

∨, 'or': **disjunction**: p ∨ q is false when both *p* and *q* are false and it is true otherwise.

p → q, 'p implies q': p → q is false only when p is true and q is false, it is true otherwise. p is called the antecedent or premise and q the conclusion or consequent.

The **contrapositive** of p → q is (~q) → (~p).

The **converse** of p → g is q → p.

The **inverse** of p → q is ~p → ~q.

Equivalent: Two propositions have the same truth value.

Biconditional: $p \leftrightarrow q$:'p if and only if q' $p \leftrightarrow q$ is true when $p \rightarrow q$ and $q \rightarrow p$ are true. $p \leftrightarrow q$ is false otherwise.

A tautology is a compound statement that is always true.

A contradiction is a compound statement that is always false.

A contingency is a compound statement that is neither a tautology nor a contradiction.

Truth tables

In a truth table

i)the first row consists of the statements

ii)the following rows give the truth value of the

statements

p	q	~q	p ∧ q	p ∨ q	p→ q	p ↔q
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Section 2 – The real number system

Real Numbers (ℝ)

Whole numbers (

Rational numbers (ℚ)

Natural numbers (ℕ)

Irrational numbers (ℤ)

Binary operations

A binary operation * on a non-empty set A is a function *: A × A → A.

Closure: * is closed with respect to A if ∀ *a*, *b* ∈ A, a * b ∈ A. Addition and multiplication are closed with respect to the set of real numbers.

Commutativity: Let *a*, *b* ∈ A. * is commutative if *a* * *b* = *b* * *a*.

Associativity: Let *a*, *b*, *c* ∈ A. * is associative if *a* * (*b** *c*) = (*a* * *b*) * *c*.

Distributivity: Let *a*, *b*, *c* ∈ A. For any two binary operations * and Δ, * distributes over Δ if a* (b Δ c) = (a* b) Δ (a * c).

Multiplication distributes over addition for real numbers.

Identity: Let *e*, *a* ∈ A. *e* is the identity element in A if *e* * *a* = *a* * *e* = *a*. The identity element is unique to the set.

The identity for addition is 0 and the identity for multiplication is 1.

Inverse: Let *a*, *b* ∈ A. *a* is the inverse of *b* and *b* the inverse of *a* if *a* * *b* = *b** *a* = *e*.

For any element *a* with respect to addition of real numbers, the inverse of *a* is -*a* since

a + (-a) = (-a) + (a) = 0.

For multiplication of real numbers, the inverse of a is $\frac{1}{a}$ except when a = 0.

The multiplicative inverse of 0 does not exist.

Mathematical proofs

Proof by exhaustion: Prove a statement is true for every value.

Direct proof: A flow of implication beginning with *P* and ending with Q.
We assume that *P* is true and show that Q must be true.

Proof by contradiction: Assume a statement is possible and reach a contradiction.

Proof by counterexample: Given a universal statement, find a **single statement** for which the universal statement is **not true** and you will disprove the universal statement.
You can disprove something by finding a single counter example, but you cannot prove something by finding one example.

Section 3 – Principle of mathematical induction

Sequences

Series: sum of the terms of a sequence

i. $\sum_{r=1}^{n} r$: sum all numbers 1 to *n*

ii. $\sum_{r=1}^{n} c$: nc

iii. $\sum cu_r$ = c $\sum u_r$

iv. $\sum (u_r + v_r) = \sum u_r + \sum v_r$

v. $\sum_{r=1}^{n} r = \frac{n\,(n+1)}{2}$

vi. $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

vii. $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$

Proof by mathematical induction

- Prove the statement true for n = 1
- Assume the statement true for n = k
- Prove the statement true for n = k + 1
- Hence, by mathematical induction, the statement is true for all *n*
- Application of proof by mathematical induction: i)Series ii)Divisibility tests

Section 4 - Polynomials

Degree of a polynomial

Remainder theorem:

When a polynomial f(x) is divided by x - ʎ, the remainder is f(ʎ).

Factor theorem:

If x - ʎ is a factor of f(x), then f(ʎ) = 0.

$x^2 - y^2 = (x - y)(x + y)$

$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

$x^4 - y^4 = (x - y)(x + y)(x^2 + y^2)$

$x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$

$x^5 + y^5 = (x+y)(x^4 - x^3y + x^2y^2 - xy^3 + y^{4)}$

$x^6 - y^6 = (x - y)(x + y)(x^2 + xy + y^2)(x^2 - xy + y^2)$

Section 5 – Indices, Surds and Logarithms

Indices

$\sqrt{xy} = \sqrt{x}\sqrt{y}$

$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$

$a^m \times a^n = a^{m+n}$

$\frac{a^m}{a^n} = a^{m-n}$

$a^{-m} = \frac{1}{a^m}$

$a^0 = 1$

$(a^m)^n = a^{mn}$

$a^{\frac{1}{m}} = \sqrt[m]{a}$

$a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = (a^{\frac{1}{n}})^m$

$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

Surds

$y = a^x$, $a > 1$

$y = a^x$, $0 < a < 1$

$y = \log_e x = \ln x$

$\ln 1 = 0$

$\ln e = 1$

$\ln x + \ln y = \ln xy$

$\ln x - \ln y = \ln \left(\frac{x}{y}\right)$

$\ln x^n = n\ln x$

$\log_a x = \frac{\log_b x}{\log_b a}$

$e^{\ln f(x)} = f(x)$

$\text{In } e^{f(x)} = f(x)$

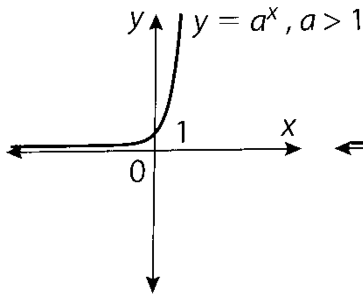
Logarithms

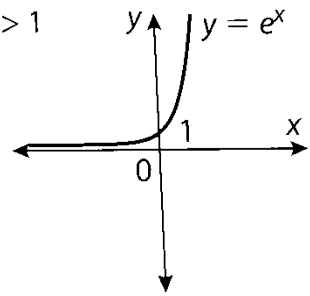
$y = a^x, a > 1$

Domain is $x \in \mathbb{R}$, Range: $y > 0$

$y = 0$ is a horizontal asymptote

a^x is an increasing function and is one-to-one



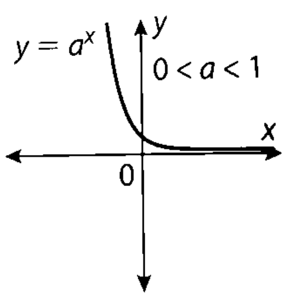


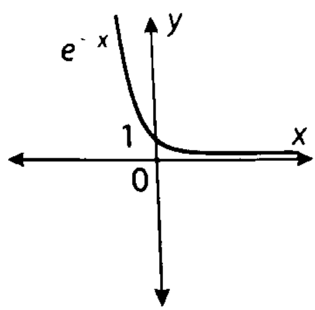
$y=a^x$, $0 < a < 1$

Domain is $x \in \mathbb{R}$, Range: $y > 0$

$y = 0$ is a horizontal asymptote

a^x is a decreasing function and is one-to-one





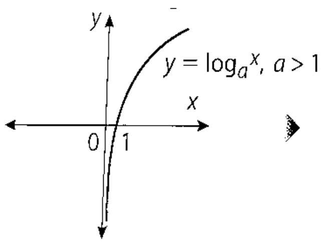
$x = a^y \leftrightarrow y = \log_a x$

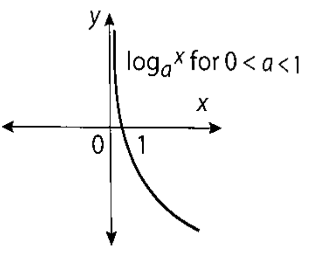
$y = \log_a x$

Domain is $x \in \mathbb{R}^+$, Range: $y \in \mathbb{R}$

y-axis is a vertical asymptote

The function is decreasing if $0 < a < 1$ and increasing if $a > 1$.





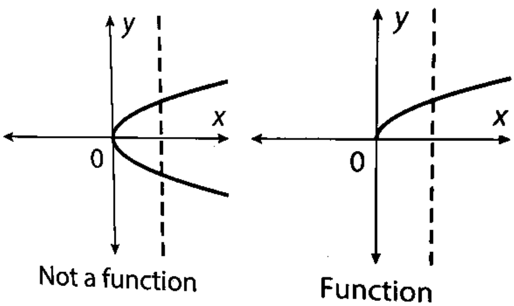
Section 6 - Functions

A function *f* from a set *A* to a set *B* assigns to each *a* ∈ *A* a single element f(a) in set B. The element in set *B* is called the image of a under *f.* The set ***A*** is called the **domain** of the function, the set ***B*** is called the **codomain** of the function. The **range** of the function is the set of elements that are the **images of *a* ∈ *A***.

A function can be described:

i. using an arrow diagram
ii. as a set of ordered pairs
iii. using a graph
iv. by a formula
v. by listing values.

Vertical line test: If a line drawn parallel to the y-axis cuts the graph at most once, the relation is a function.



One-to-one (injective): A function f: X→Y is injective if every element of Y is mapped onto by one and only one element of X. **(No two x-values can have the same y-image).**

To show that f(x) is one-to-one:

Method 1: If f(a) = f(b) → a = b, then f(x) is one-to-one.

Method 2: A line drawn parallel to the x-axis on the graph of f(x) must cut f(x) at most once.

Onto (surjective): A function f: X → Y is surjective if every y is mapped onto by at least one x.

For a surjective function the codomain and the range must be the same.

To show that f(x) is onto: a line drawn parallel to the x-axis must cut f(x) at least once if f(x) is onto.

Bijective: A function f: X → Y is bijective if it is one-to-one and onto.

Inverse functions: The inverse of f(x) is denoted by $f^{-1}(x)$. $f^{-1}(x)$ exists if f(x) is one-to-one.

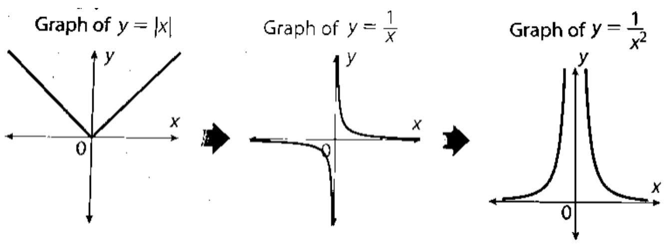
Domain of f(x) = range of $f^{-1}(x)$

Range of f(x) = domain of $f^{-1}(x)$

To sketch $f^{-1}(x)$: reflect f(x) in the line y = x.

$ff^{-1}(x) = F^{-1} f(x) = x$

For gf(x), the range of *f* is the domain of *g*.



If $\frac{f(b)-f(a)}{b-a} > 0$ for b > a, then f(x) is increasing in (a,b).

If $\frac{f(b)-f(a)}{b-a} < 0$ for b > a, then f(x) is a decreasing in (a,b).

Sketching graphs from y = f(x):

(i) *f(ax)*: stretch along the x-axis by factor $\frac{1}{a}$

(ii) *af(x)*: stretch along the y-axis by factor *a*.

(iii) *f(x + a):* shift to the left by *a* units if *a* is positive and a shift to the right by a units if *a* is negative.

(iv) *f(x) + a*: shift upwards by *a* units if *a* is positive and a shift downwards by *a* units if *a* is negative.

(v) *f(-x)*: reflect f(x) in the y-axis.

(vi) -*f(x)*: reflect f(x) in the x-axis.

(vii) *f(x):* reflect anything below the x-axis to above the x-axis, the section of the curve above the x-axis remains as it is.

Section 7 – Cubic Polynomials

Ouadratics

Let a, ß be the roots of the equation

$ax^2 + bx + c = 0$

$a + ß = \frac{-b}{a}$, $a ß = \frac{c}{a}$

Given the sum of the roots and the product of the roots the equation is:

x^2 - (sum of the roots) x + (product of the roots) = 0

$a^2 + ß2 = (a + ß)^2 – 2aß$

$a^2 - ß2 = (a - ß)(a + ß)$

$(a - ß)^2 = (a + ß)^2 - 4aß$

If $a > ß$, $(a - ß)$ is positive

Cubics

Let a, ß, y be the roots of

$ax^3 + bx^2 + cx + d = 0$

$a + ß + y = \frac{-b}{a}$

$aß + ßy + ya = \frac{c}{a}$

$aßy = -\frac{d}{a}$

$a^2 + ß^2 + y^2 = (a + ß + y)^2 - 2(aß + ß y + ya)$

or $\sum a^2 = (\sum a)^2 - 2\sum a ß$

$\sum a^3 - (\sum a)^3 - 3 (\sum a) \sum (aß) + 3aßy$

The cubic equation is:

x^3- (sum of the roots) x^2 + (sum of the product of

two of the roots at a time) x- (product of the roots)

= 0

Section 8 – Inequalities and the modulus function

Properties of inequalities

If a > b, then a + c > b + c

If a > b and c > 0, then ac > bc

If a > b and c < 0, then ac < bc

If a > b, then $\frac{1}{a} < \frac{1}{b}$

Solving inequalities using a sign table

i. Make the function f(x) > 0 or f(x) < 0

ii. Identify the critical values of f(x).

iii. Draw up a sign table using the critical values and the factors of the function.

iv. Identify the solution set.

Modulus function

$$|x| = \begin{pmatrix} x \; if \; x > 0 \\ 0 \; if \; x = 0 \\ -x \; if \; x < 0 \end{pmatrix}$$

$|x|^2 = x^2$

$|xy| = |x||y|$

$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

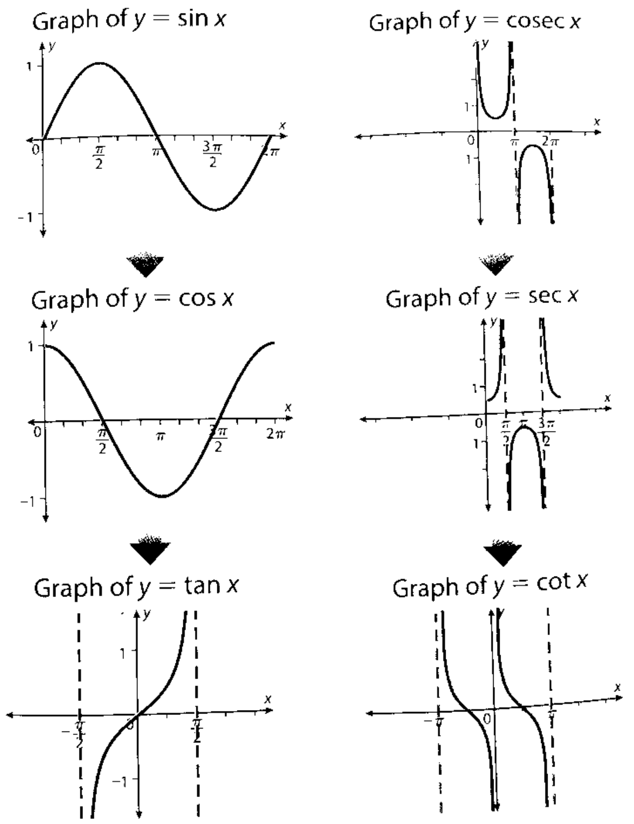
$|x| < |y|$ if and only if $x^2 < y2$

$|x| > c$ if and only if $x > c$ or $x < -c$.

$|x| < c$, where c is a constant if and only if $–c< x< c$.

Section 9 - Trigonometry

<u>Graphs and general solutions</u>



General solution of sin x: x = $n\pi + (-1)^n a, n \in \mathbb{Z}$

General solution of cos x: x = $2n\pi \pm a, n \in \mathbb{Z}$

General solution of tan x: x= $n\pi + a, n \in \mathbb{Z}$

Trigonometric identities

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \pm \sin A \sin B$

$\tan(A \pm B) =. \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin 2\theta = 2 \sin\theta \cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$

If t = tan $\frac{\theta}{2}$, then:

sin θ = $\frac{2t}{t^2 + 1}$

cos θ= $\frac{1 - t^2}{1 + t^2}$

tan θ= $\frac{2t}{1 - t^2}$

2sin A cos B = sin (A + B) + sin (A - B)

2 cos A sin B = sin (A + B) - sin (A - B)

2 cos A cos B = cos (A + B) + cos (A - B)

-2sin A sin B = cos (A + B) - cos (A - B)

sin A + sin B = 2sin $\frac{A+B}{2}$A + B cos $\frac{A-B}{2}$A - E

sin A - sin B = 2cos $\frac{A+B}{2}$ sin $\frac{A-B}{2}$

cos A + cos B = 2cos $\frac{A+B}{2}$ cos $\frac{A-B}{2}$

cos A – cos B = -2sin $\frac{A+B}{2}$ sin $\frac{A-B}{2}$

- a cos θ + b sin θ = r cos θ

  where r > 0 and 0° < a < 90°
- Maximum (a cos θ + b sin θ) = r,

  when cos (θ - a) = 1
- Minimum (a cos θ + b sin θ) = -r,

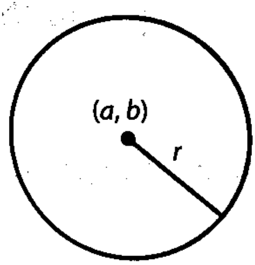
  when cos (θ - a) = -1

Section 10 – Coordinate geometry

Circles

- A circle is the locus of a point which moves in a plane so that it is equidistant from a fixed point.
- Equation of a circle centre (0, 0) and radius r: $x^2+y^2=r^2$
- Equation of a circle centre (a, b) and radius r: $(x-a)^2 + (y - b)^2 = r^2$
- General equation of a circle:

  $Ax^2 + By^2 + Cx + Dy + E = 0$, $A = B$



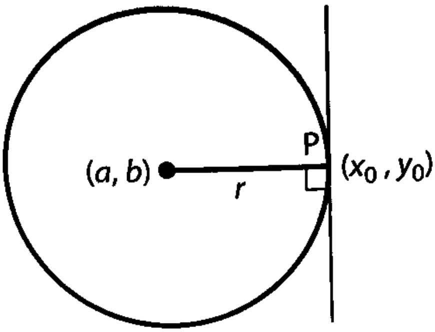
- Parametric equation of a circle centre (a, b) and radius r:

  $x = a + r\cos\theta \qquad y = b + r\sin\theta$

  or

  $x = a + r\sin\theta \qquad y = b + r\cos\theta$

- Gradient of the normal at P = $\frac{y_0 - b}{x_0 - a}$



Since the tangent is perpendicular to the radius, the gradient of the tangent of P is $-\frac{y_0 - b}{x_0 - a}$

- Given three points on a circle we can find its equation by using the general form

  $(x-a)^2 + (y- b)^2 = r^2$.

  We form three equations and solve them simultaneously to find *a*, *b* and *r*.

- Or, if the circle passes through P, Q and R, the centre of the circle is the point of intersection of the perpendicular bisectors of PQ and QR. The radius can be found using the centre and any of the points P, Q and R.

Ellipses

- An ellispse is the locus of points, the sum of whose distance from two fixed points is constant. The two fixed points are the **foci** of the ellipse.
- Equation of an ellipse with centre (0, 0) and foci at (c, 0) and (-c, 0) is: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- Equation of an ellipse with foci (0, c) and (0, -c) is: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- Equation of an ellipse with centre (h, k) and major axis parallel to the x-axis is:

  $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

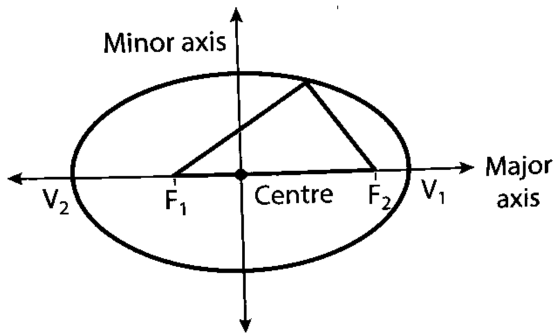
  Foci are at (h + c, k) and (h - c, k).
- Equation of an ellipse with centre (h, k) and major axis parallel to the y-axis is:

  $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$

  Foci are at (h, k + c) and (h, k - c).
- General equation of an ellipse:

  Ax2 + B2 + Cx + Dy + E = 0, A ≠ B

- Graph of an ellipse centre C, foci F_1, and F_2



- Parametric equations of an ellipse centre (h, k) and radius r:

  x = h + a cos t y = k+ b sin t

- Gradient of the tangent at P (a cos t, b sin t) is $\frac{-b\cos t}{a\sin t}$

  and the gradient of the normal is $\frac{a\sin t}{b\cos t}$

- Equation of the tangent can be found by using the gradient and point.

Parabolas

- A parabola is a set of all points in a plane, which are at the same distance from a fixed-point P as they are from a fixed line *I*. P is called the focus and *I* is called the directrix.
- Equation of a parabola with centre (0, 0) and opening upwards is: $x^2 = 4ay$
- Equation of a parabola with centre (0, 0) and opening to the right is: $y^2 = 4ax$
- Equation of a parabola with axis of symmetry parallel to the y-axis is:
  $y = ax^2 + bx + c \quad a \neq 0$
- Equation of a parabola with axis of symmetry parallel to the x-axis is:
  $x = ay^2 + by + c \quad a \neq 0$
- Equation of a parabola in vertex form:
  $y = a(x - h)^2 + k$
  or
  $y = a(y-k)^2 + h$

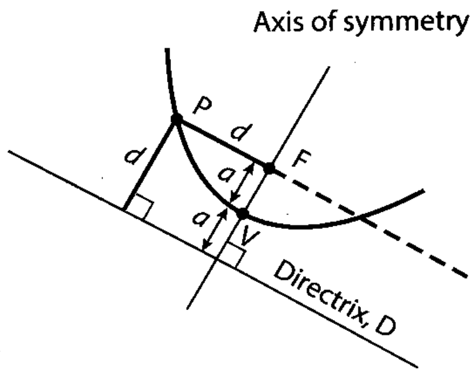
- Equation of a parabola in conics form:

  $4p (y - k) = (x - h)^2$

  or

  $4p (x - h) = (y-k)^2$

where p is the distance from the vertex to the focus (also the distance from the vertex to the directrix). (2*p* is the distance from the focus to the directrix).



- Parametric equations of a parabola are:

  $x = at^2 \qquad y = 2at \quad t \in \mathbb{R}, \; a > 0$

- Gradient of the tangent to a parabola at P $(at^2, 2at)$ is $\frac{1}{t}$

- Gradient of the normal to a parabola at P $(at^2, 2at)$ is $- t$.

Section 11 – Vectors in three dimensions

Let **u** = $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and **v** = $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

u + **v** = $\begin{pmatrix} u_1 & + & v_1 \\ u_2 & + & v_2 \\ u_3 & + & v_3 \end{pmatrix}$

u – **v** = $\begin{pmatrix} u_1 & - & v_1 \\ u_2 & - & v_2 \\ u_3 & - & v_3 \end{pmatrix}$

λ = $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \lambda u_1 \\ \lambda u_2 \\ \lambda u_3 \end{pmatrix}$, $\lambda \in \mathbb{R}$

u = v iff $u_1 = v_1$, $u_2 = v_2$, $u_3 = v_3$

u • v = $u_1v_1 + u_2v_2 + u_3v_3$

$|u| = \sqrt{u_1^2 + u_2^2 + u_3^2}$

A unit vector is a vector with magnitude 1 unit

A unit vector in the direction of **u** is $\frac{\boldsymbol{u}}{|\boldsymbol{u}|}$

u • v = $|u||v|\cos\theta$, where θ is the angle between **u** and **v**.

u and **v** are perpendicular iff **u • v** = 0.

u and **v** are parallel iff one is a scalar multiple of the other.

$$i = \begin{pmatrix}1\\0\\0\end{pmatrix},\ j= \begin{pmatrix}0\\1\\0\end{pmatrix},\ k= \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}a\\b\\c\end{pmatrix} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Lines

Let **a** be a point on a line and **m** be the direction vector. The equation of the line is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{m}$, $\lambda \in \mathbb{R}$

Let **p** and **q** be two points on a line.

The equation of the line is $\mathbf{r}= \mathbf{p}+\lambda\ (\mathbf{q}-\mathbf{p})$, $\lambda \in \mathbb{R}$

Vector equation of a line:

$$\mathbf{r} = \begin{pmatrix}a\\b\\c\end{pmatrix} + \lambda \begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix},\ \lambda \in \mathbb{R}$$

Cartesian equation of a line:

$$\frac{x-a}{x_0} = \frac{y-b}{y_0} = \frac{z-c}{z_0}$$

Parametric equations of a line:

$$\lambda \in \mathbb{R} \begin{cases} x = a + \lambda x_0 \\ y = b + \lambda y_0 \\ z = c + \lambda z_0 \end{cases}$$

X= a + 1xo y=b+1yo < 1 € R

Skew lines are lines that are not parallel and do not intersect.

The angle between two lines is the angle between their direction vectors.

Two lines are perpendicular iff the scalar product of their direction vectors is 0.

Planes

Let $\hat{\boldsymbol{n}}$ be a unit vector perpendicular to a plane and *d* be the distance from the origin to the plane. The equation of the plane is: r · $\hat{\boldsymbol{n}}$ = *d*

Let **a** be a point on a plane and **n** a vector perpendicular to the plane. The equation of the plane is: **r · n = a· n**

Vector equation of a plane:

$$\mathbf{r}\ .\begin{pmatrix} A \\ \boldsymbol{B} \\ \boldsymbol{C} \end{pmatrix} \cdot \mathbf{D}$$

Cartesian equation of a plane:

Ax + By + Cz = D

Where $\begin{pmatrix} A \\ \boldsymbol{B} \\ \boldsymbol{C} \end{pmatrix}$ is a vector perpendicular to the plane.

Section 12 – Limits and continuity

Limits

Exists:

$\lim_{x \to b} c$ = c, where c is a constant.

$\lim_{x \to b} x^n$ = b^n where b is a positive integer.

$\lim_{x \to b}(f(x) + g(x)) = \lim_{x \to b} f(x) + \lim_{x \to b} g(x)$

$\lim_{x \to b}(cf(x)) = c \lim_{x \to b} f(x)$, where c is a constant.

$\lim_{x \to b} f(x)\, g(x) = \lim_{x \to b} f(x) \times \lim_{x \to b} g(x)$

$\lim_{x \to b} \frac{f(x)}{g(x)} = \frac{\lim_{x \to b} f(x)}{\lim_{x \to b} g(x)}$ Provided that $\lim_{x \to b} g(x) \neq 0$.

$\lim_{x \to b}(f(x))^n = \left(\lim_{x \to b} f(x)\right)^n$

$\lim_{x \to b} \sqrt{f(x)} = \sqrt{\lim_{x \to b}(f(x))}$

Does not exist:

A limit does not exist, if:

a) the left-hand limit ≠ right-hand limit

b) f(x) oscillates infinitely

c) f(x) increases or decreases infinitely

Methods of finding limits:

- Direct substitution
- Factorizing
- Rationalizing
- Limits tending to infinity: treat differently

Special limits

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$$

Continuity

Continuous:

f(x) is continuous at x = a, if:

a) $\lim_{x \to a} f(x) = l_1$

b) f(a) = l_2

c) $l_1 = l_2$

Discontinuous:

Removable - Point

Non-removable – Jump Infinite

<u>**Section 13 – Differentiation**</u>

Notation for differentiation

f'(x), $\frac{dy}{dx}$, f'(x), y', $\frac{dy}{dx}$ (f(x))

Definition:

f'(x) = $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$\boldsymbol{\frac{d}{dx}}$ **$(\boldsymbol{x^n})$ =** nx^{n-1}

$\frac{d}{dx}(ax^n)$ = anx^{n-1}, where *a* is a constant

$\frac{d}{dx}((ax+b)^n)$ = $an(ax+b)^{n-1}$, where *a* is a constant

$\frac{d}{dx}$(sin x) = cos x

$\frac{d}{dx}$ (cos x) = -sin x

$\frac{d}{dx}$ (tan x) = $\sec^2$ x

$\frac{d}{dx}$ (sec x) = sec x tan x

$\frac{d}{dx}$ (cosec x) = -cosec x cot x

$\frac{d}{dx}$ (cot x) = $-\text{cosec}^2$ x

Standard results

Chain rule:

$\frac{d}{dx}$ (fg(x)) = f(g'(x)) × g'(x)

Product rule:

y=*uv*, where *u* and *v* are functions of x

$\rightarrow \frac{dy}{dx}$ = u $\frac{dv}{dx}$ + $\frac{du}{dx}$

Quotient rule:

y= $\frac{u}{v}$, where *u* and *v* are functions of x

$\rightarrow \boldsymbol{\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$

Section 14 – Application of differentiation

Tangents and normals

a) Gradient of a tangent = $\frac{dy}{dx}$

b) Gradient of a normal = $\frac{-1}{\frac{dy}{dx}}$

Rate of change

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

Curve sketching

Look for:

- intercepts
- turning points
- asymptotes

Asymptotes occur on rational functions f(x) = $\frac{P(x)}{Q(x)}$

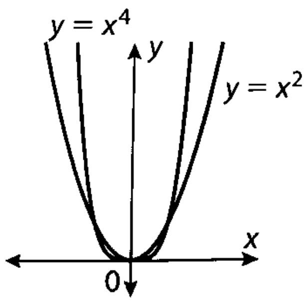
Vertical asymptotes occur at Q(x) = 0

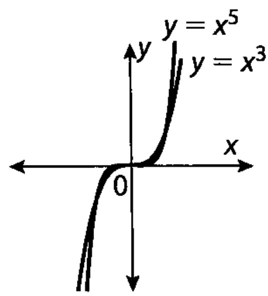
Horizontal asymptotes occur at $\lim_{x \to \infty}$ f(x) = l

→ y = l is a horizontal asymptote.

Graph of a polynomial

i. Find where the curve cuts the x- and y-axis if possible.
ii. Find values of y for values of x between the points where the graph cut the x-axis.
iii. Draw a smooth, continuous curve.





Increasing and decreasing functions

A function is increasing over an interval if $\frac{dy}{dx} > 0$ over the interval.

A function is decreasing over an interval if $\frac{dy}{dx} < 0$ over the interval.

Stationary points

Stationary points exist at $\frac{dy}{dx} = 0$

<u>First derivative test for classification of x = a</u>

If $\frac{dy}{dx} < 0$ for $x < a$ or $\frac{dy}{dx} > 0$for $x > a$, then there is a minimum point when x = a.

If $\frac{dy}{dx} > 0$ for $x < a$ or $\frac{dy}{dx} < 0$ for $x > a$, then there is a maximum point when $x = a$.

If $\frac{dy}{dx}$ does not change sign as it passes through a, then there is a point of inflexion when $x = a$.

<u>Second derivative test for classification of x = a</u>

If $\frac{d^2y}{dx^2} > 0$ when $x = a$, then there is a minimum point at $x = a$.

If $\frac{d^2y}{dx^2} < 0$ when $x = a$, then there is a maximum point at $x = a$.

If $\frac{d^2y}{dx^2} = 0$, you must test further (in this case you can go back to the first derivative test).

Points of inflexion occur at $\frac{d^2y}{dx^2} = 0$. (You must check concavity since this is not a necessary condition.)

Section 15 – Integration

Integration is the reverse of differentiation

$\int cf(x)\, dx = c\int f(x)\, dx$

$\int f(x) \pm g(x)\, dx = \int f(x) dx \pm g(x)\, dx$

$\int_a^b f(x)\, dx = F(b) - F(a)$

$\int_a^b f(x)\, dx = \int_a^b f(t)\, dt$

$\int_0^a f(x)\, dx \int_0^a f(a-x)\, dx,\ a > 0$

$\int_a^b f(x)\, dx = - \int_a^b f(x)\, dx$

Some standard results

$\int x^n\, dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$

$\int (ax + b)^n\, dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c,\ n \neq -1$

$\int \sin x\ dx = - \cos x + c$

$\int \cos x\, dx = \sin x + c$

$\int \sec^2 x\, dx = \tan x + c$

$\int \operatorname{cosec}^2 x\, dx = -\cot x + c$

$\int \sin^n x \cos x\, dx = \frac{\sin^{n+1} x}{n+1} + c,\ n \neq -1$

$\int \cos^n x \sin x\, dx = \frac{\cos^{n+1} x}{n+1} + c,\ n \neq -1$

$\int \tan^n x \sec^2 x\, dx = \frac{\tan^{n+1} x}{n+1} + c,\ n \neq -1$

$\int \sin^2 x\, dx = \frac{1}{2} \int 1 - \cos 2x\, dx$

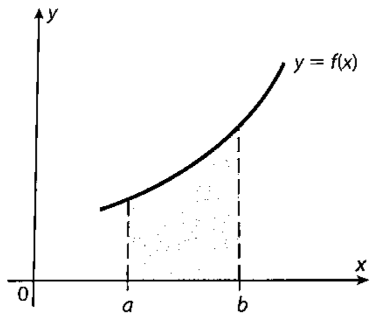
$\int \cos^2 x\, dx = \frac{1}{2} \int 1 + \cos 2x\, dx$

$\int \tan^2 x\, dx = \int \sec^2 x - 1\, dx$

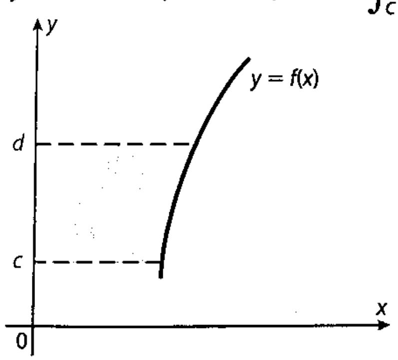
Section 16 – Applications of Integration

Area under a curve

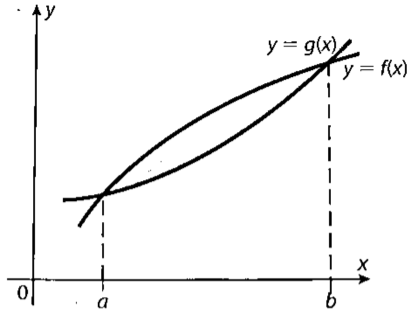
Area between the curve and the x-axis from $x = a$ to $x = a$ is $\int_a^b y\, dx$



Area between the curve and the y-axis from $y = c$ to $y = d$ is $\int_c^d x\, dx$



Area between two curves

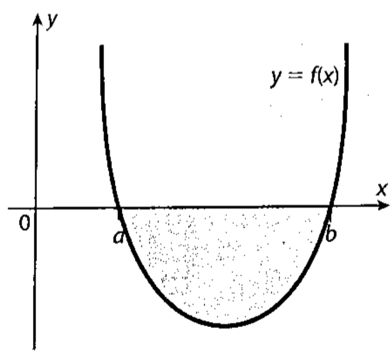


(i) Find the points of intersection of the two curves.

(ii) Shaded area =

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

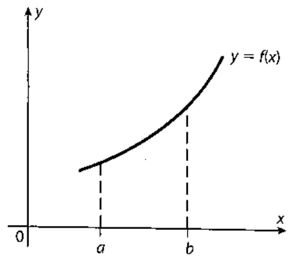
Area below the x-axis



$$\text{Shaded area} = \left| \int_a^b f(x)\,dx \right|$$

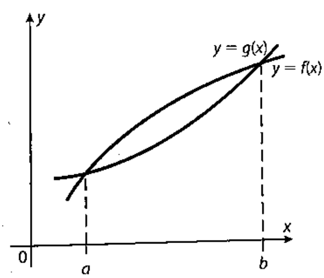
Rotation about the x-axis

Volume of the solid formed when the shaded region is rotated 2π radians about the x-axis is $\pi \int_a^b y^2 \, dx$



Volume of the solid formed when the shaded region is rotated 2π radians about the x-axis is

$$\pi \int_a^b (g(x))^2 \, dx - \pi \int_a^b (f(x))^2 \, dx$$



Rotation about the y-axis

Volume of the solid formed when the shaded region is rotated 2π radians about the y-axis is $\pi \int_c^d x^2 \, dy$

Section 17 – Differential equations

i. First order differential equations:
$\frac{dy}{dx}$ = f(x) g(y)

ii. Separate the variables: $\frac{1}{g(y)}$ dx = f(x) dx

iii. Integrate both sides: $\int \frac{1}{g(y)}$ dy = $\int f(x)\,dx$

iv. Do not forget to add the constant of integration.

v. If *x* and *y* are given, use these values to find the constant.

---

a. Worded equations
b. Identify the variables clearly.
c. Remember a differential equation represents a rate of change.
d. One variable will be changing with respect to the other: this gives the rate of change.
e. For example: *x* changes proportional to *t* gives: $\frac{dx}{dt}$ = kt
f. Follow the steps for solving first order differential equations.

$\frac{d^2y}{d^2x}$ = f(x)

Integrate twice to solve. At each stage introduce a constant of integration.