MODULE 1 : COMPLEX NUMBERS

If we were to use the quadratic formula to solve $5x^2 + 6x + 10 = 0$, the result would be the following:

$$x = \frac{-(-6) \pm \sqrt{(6)^2 - 4(5)(10)}}{2(5)}$$

$$x = \frac{-6 \pm \sqrt{-164}}{10}$$

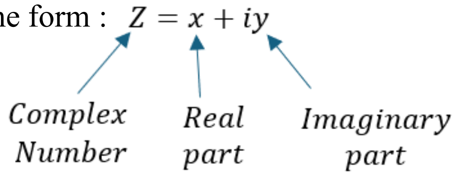
Since the square root of a negative number cannot be found, we factor out $\sqrt{-1}$.

In mathematics, this imaginary number is denoted by i.

$\therefore i = \sqrt{-1}$

This allows us to obtain complex numbers.

Complex numbers are generally of the form : $Z = x + iy$



N.B.

- $i = \sqrt{-1}$
- $i^2 = -1$
- $i^3 = -i$
- $i^4 = 1$
- $i^5 = i$

<u>ADDITION, SUBTRACTION AND MULTIPLICATION OF COMPLEX NUMBERS</u>

Example: Given that $Z_1 = 2 + 2i$ and $Z_2 = 4 - 3i$, then

- $\boldsymbol{Z_1 + Z_2 = (2 + 2i) + (4 - 3i)}$

  Add the real and imaginary separately

  $= (2 + 4) + (2i + (-3i))$

  $= 6 - i$

- $\boldsymbol{Z_1 - Z_2 = (2 + 2i) - (4 - 3i)}$

  Subtract the real and imaginary separately

  $= (2 - 4) - (2i - (-3i))$

  $= -2 - 5i$

- $\boldsymbol{Z_1 \cdot Z_2 = (2+2i)\,(4-3i)}$

  Expand Brackets

  $= (2)(4) + (2)(-3i) + (2i)(4) + (2i)(-3i)$

  $= 8 - 6i + 8i - 6i^2$ Since $i^2 = -1$

  $= 8 + 2i - 6(-1)$

  $= 8 + 6 + 2i$

  $= 14 + 2i$

CONJUGATE OF Z

If $Z = x + iy$ then the conjugate of Z is : $\bar{Z} = x - iy$.

N.B.

$Z \cdot \bar{Z} = x^2 + y^2$ (This a real number since there are no imaginary parts)

DIVISION OF COMPLEX NUMBERS

Rule : $\dfrac{Z_1}{Z_2} = \dfrac{Z_1}{Z_2} \times \dfrac{\bar{Z}_2}{Z_2}$ ⟵ conjugate of Z_2

SQUARE ROOT OF A COMPLEX NUMBER

Example: Find the square root of $2 + 3i$

1. **Let $\boldsymbol{\sqrt{Complex\ no.} = Z}$**

   $\sqrt{-15 + 8i} = Z$

2. **Replace $\boldsymbol{Z = x + iy}$**

   $\sqrt{-15 + 8i} = x + iy$

3. **Square both sides**

   $-15 + 8i = (x + iy)^2$

4. **Expand and simplify**

   $2 + 3i = (x + iy)\,(x + iy)$

   $2 + 3i = x^2 + xiy + xiy + i^2y^2$

$2 + 3i = x^2 + 2xiy + (-1)y^2$

$-15 + 8i = x^2 - y^2 + 2xiy$

5. Compare both real and imaginary parts

6. Make y the subject of the formula in the imaginary part

- Real : $-15 = x^2 - y^2$.....eq 1
- Imaginary : $8 = 2xy \longrightarrow y = \frac{4}{x}$ .....eq 2

7. Solve Simultaneously : Substitute eq 2 into eq 1

$x^2 - (\frac{4}{x})^2 = -15$

$x^2 - \frac{16}{x^2} = -15$ Multiply throughout by x^2

$x^4 - 16 = -15x^2$

$x^4 + 15x^2 - 16 = 0$ Let $y = x^2$ and solve using factorization or the quadratic formula

$y^2 + 15y - 16 = 0$

$(y - 1)\ (y + 16) = 0$

- $y - 1 = 0$

  $y = 1$

  $x^2 = 1$

  $x = \sqrt{1} = \pm 1$

- $y + 16 = 0$

  $y = -16$

  $x^2 = -16$

  No real solution

∴ **Substitute $x = \pm 1$ into eq 2 to find the values of y**

$y = \frac{4}{\pm 1} = \pm 4$

8. Write in the form $Z = x + iy$

$Z = \pm(1 + 4i)$

SOLVING COMPLEX QUADRATICS

To solve a complex quadratic of the form $az^2 + bz + c = 0$, where a, b and c are complex (or at least one is complex) we use:

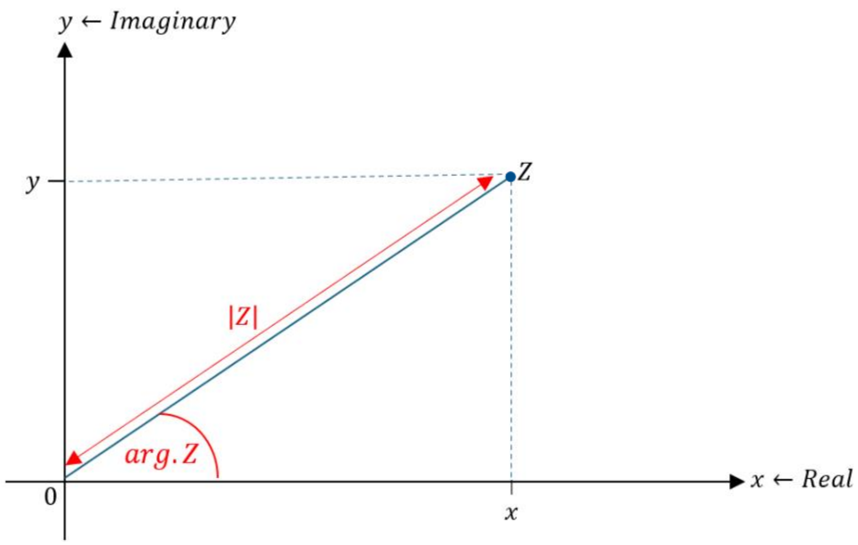
$$Z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To simplify the above equation, we generally have to find the square root of a complex number.

ARGAND DIAGRAMS

If we have a complex number, it can be represented by an argand diagram.
For this we need to obtain the following:

- $|Z|$ = modulus or magnitude of Z
- $arg.Z$ = argument of Z



- **Modulus :** When $Z = x + iy$

  $$|Z| = \sqrt{x^2 + y^2}$$

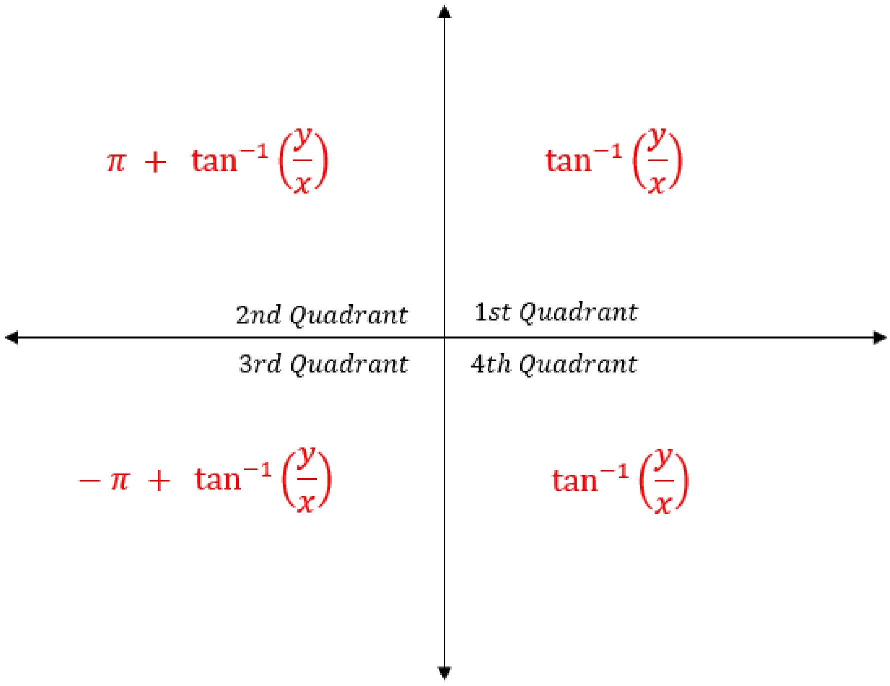
- **Argument :** When $Z = x + iy$

  $$arg.Z = \tan^{-1}\left(\frac{y}{x}\right)$$

  N.B. $-\pi \leq arg.Z \leq \pi$ (work in radians)

  When the argument is not in the range, find its equivalent using a graph sketch.

Generally, to find $arg.Z$ in each quadrant, we use:



RULES

- $\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|}$

  $arg.\left(\frac{Z_1}{Z_2}\right) = arg.Z_1 - arg.Z_2$

- $|Z_1 \cdot Z_2| = |Z_1| \cdot |Z_2|$

  $arg.(Z_1 \cdot Z_2) = arg.Z_1 + arg.Z_2$

FORMS OF COMPLEX NUMBERS

- Cartesian Form : $\boldsymbol{Z = x + iy}$
- Polar Form
  - $\boldsymbol{Z = r\,(Cos\theta + i\,Sin\theta)}$
  - $\boldsymbol{Z^n = r^n\,(\mathrm{Cos}\,n\theta + i\,Sin\,n\theta)}$
- Exponential Form
  - $\boldsymbol{Z = re^{i\theta}}$
  - $\boldsymbol{Z^n = r^n e^{n\theta.\,i}}$

$Where$
$r = modulus\ of\ Z$
$\theta = argument\ of\ Z$

BINOMIAL EXPANSION

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} \cdot b^1 + \binom{n}{2} a^{n-2} \cdot b^2 + \cdots + \binom{n}{n} b^n$$

Binomial expansion helps us to use De Moivre's Theorem.

DE MOIVRE'S THEOREM

De Moivre's Theorem states that $(Cos\theta + i\ Sin\theta\ )^n$ is equivalent to $(\text{Cos}\ n\theta + i\ Sin\ n\theta)$.

N.B. *When a question asks to obtain your answer in terms of $Cos\theta$ or in terms of $Sin\theta$ only, substitute appropriate trig identities and simplify.*

LOCUS OF A COMPLEX NUMBER

- **Circle**

  For a circle in the complex plane, the general equation is: $|\boldsymbol{Z} - \boldsymbol{c}| = \boldsymbol{r}$

  Where:

  Z is the complex number $x + iy$

  c is the centre of the circle

  r is the radius of the circle

Example:

$|Z + 2 + 2i| = 3$

Substitute $Z = x + iy$

$|x + iy + 2 + 2i| = 3$

Pair the real and pair the imaginary

$|(x+2) + (y+2)i| = 3$

↑ x ↑ y

We know modulus $= \sqrt{x^2 + y^2}$

So, $\sqrt{(x+2)^2 + (y+2)^2} = 3$

Square both sides

$(x+2)^2 + (y+2)^2 = 3^2$ ← *In the form of an equation of a circle.*

∴ Circle with centre, $c\ (-2, -2)$ and radius, $r = 3$

- **Straight Line**

  For a straight line in the complex plane, the general equation is: $\boldsymbol{ax + by + c = 0}$

  Where:

  x is the real part of the complex number

  y is the imaginary part of the complex number

  a, b, c are constants

Example:

$|Z - 2 - 2i| = |Z + 3 - i|$

Substitute $Z = x + iy$

$|x + iy - 2 - 2i| = |x + iy + 3 - i|$

$|(x - 2) + (y - 2)i| = |(x + 3) + (y - 1)i|$

We know modulus $= \sqrt{x^2 + y^2}$

So, $\sqrt{(x-2)^2 + (y-2)^2} = \sqrt{(x+3)^2 + (y-1)^2}$

Square both sides

$(x-2)^2 + (y-2)^2 = (x+3)^2 + (y-1)^2$

$x^2 - 4x + 4 + y^2 - 4y + 4 = x^2 + 6x + 9 + y^2 - 2y + 1$

Simplify

$-2y = 10x + 2$

Divide by -2

$y = -5x - 1$

∴ Straight line with equation $y = -5x - 1$

- **Straight Line based on Argument**

  A straight line in a complex plane given by $arg.(Z - a) = \theta$ is a half-line starting at a and making an angle θ with the positive axis.

Example:

$arg.(Z + 2 + i) = \frac{3\pi}{4}$ ← This tells you that it should be drawn in the second quadrant.

$arg.(Z - (-2 - 1))$ ∴ half-line starts at $(-2, -1)$

$arg.(x + iy + 2 + i) = \frac{3\pi}{4}$

$arg.((x+2)+(y+1)i) = \frac{3\pi}{4}$

$\tan^{-1}\left(\frac{y}{x}\right) = \frac{3\pi}{4}$

$\tan^{-1}\left(\frac{y+1}{x+2}\right) = \frac{3\pi}{4}$

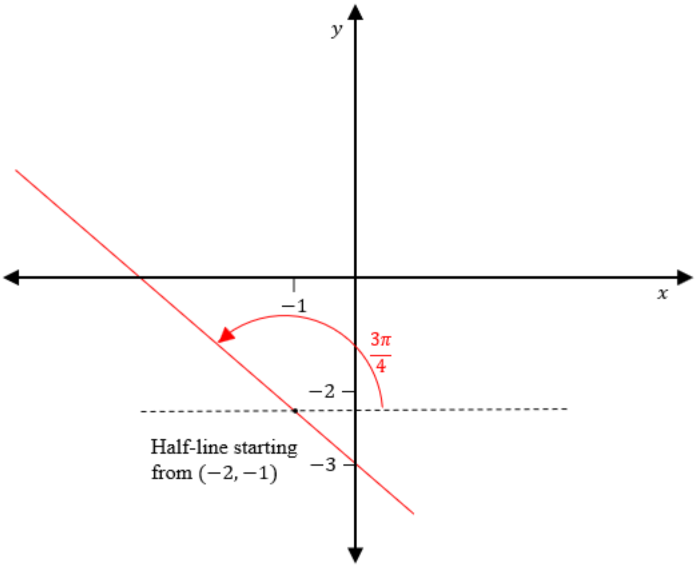
$\frac{y+1}{x+2} = \tan\frac{3\pi}{4}$

$\frac{y+1}{x+2} = -1$

$y+1 = -x-2$

$y = -x-3$

∴ Straight line with equation $y = -x-3$



MODULE 1 : DIFFERENTIATION

DIFFERENTIATION REVIEW

- $y = ax^n$
  $y' = n \cdot ax^{n-1}$

- $y = ax$
  $y' = a$

- $y = a$
  $y = 0$

- $y'' = \dfrac{d^2y}{dx^2} = \dfrac{d\left(\frac{dy}{dx}\right)}{dx}$

- $y = (ax+b)^n$
  $\dfrac{dy}{dx} = n \cdot a \cdot (ax+b)^{n-1}$ OR $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ ← Chain Rule

- Product Rule : $U\dfrac{dv}{dx} + V\dfrac{du}{dx}$

- Quotient Rule : $\dfrac{V\frac{du}{dx} - U\frac{dv}{dx}}{V^2}$

- $y = \cos x$
  $y' = -\sin x$

- $y = \sin x$
  $y' = \cos x$

- $y = \tan x$
  $y' = \sec^2 x$

DIFFERENTIAL OF $e^{f(x)}$

When $y = e^{f(x)}$

$$\frac{dy}{dx} = f'(x) \cdot e^{f(x)}$$

Example:

$y = e^{2x}$

$\frac{dy}{dx} = 2 \cdot e^{2x}$

DIFFERENTIAL OF $\ln f(x)$

When $y = \ln f(x)$

$$\frac{d\,[\ln f(x)]}{dx} = \frac{f'(x)}{f(x)}$$

Example:

$y = \ln\ |2x - 1|$

$\frac{dy}{dx} = \frac{2}{2x-1}$

> ***N.B.***
> ln cannot be negative, therefore a modulus goes on $f(x)$

The differential of e^x is e^x

Proof:

$$\frac{d(e^x)}{dx} \rightarrow y = e^x$$

So,

$ln\, y = ln\, e^x$

$ln\, y = x \cdot ln\, e$ *Remember $ln\, e = 1$*

$ln\, y = x$

$$\frac{\left(\frac{dy}{dx}\right)}{y} = 1$$

$$\frac{dy}{dx} = 1 \cdot y$$

$$\frac{dy}{dx} = 1 \cdot e^x = e^x$$

DIFFERENTIAL OF INVERSE TRIGS

When $y = \sin^{-1} x$, find $\frac{dy}{dx}$.

$y = \sin^{-1} x$

$\sin y = x \quad \rightarrow \quad x = \sin y$

$\therefore \frac{dx}{dy} = \cos y$

Since $\frac{dx}{dy} = \cos y$, then $\frac{dy}{dx} = \frac{1}{\cos y}$

We know:

$\cos^2 y + \sin^2 y = 1$

$\therefore \cos^2 y = 1 - \sin^2 y$

And $\cos y = \sqrt{1 - \sin^2 y}$

Since $x = \sin y$

then $\cos y = \sqrt{1 - x^2}$

Hence, $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

$$\frac{d(\sin^{-1} y)}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

When $y = \cos^{-1} x$, find $\frac{dy}{dx}$.

$y = \cos^{-1} x$

$\cos y = x \quad \rightarrow \quad x = \cos y$

$\therefore \frac{dx}{dy} = -\sin y$

Since $\frac{dx}{dy} = -\sin y$, then $\frac{dy}{dx} = -\frac{1}{\sin y}$

We know:

$\cos^2 y + \sin^2 y = 1$

$\therefore \sin^2 y = 1 - \cos^2 y$

And $\sin y = \sqrt{1 - \cos^2 y}$

Since $x = \cos y$

then $\sin y = \sqrt{1 - x^2}$

Hence, $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

$$\frac{d(\cos^{-1} y)}{dx} = -\frac{1}{\sqrt{1 - x^2}}$$

When $y = \tan^{-1} x$, find $\frac{dy}{dx}$.

$y = \tan^{-1} x$

$\tan y = x \quad \rightarrow \quad x = \tan y$

$\therefore \frac{dx}{dy} = \sec^2 y$

Since $\frac{dx}{dy} = \sec^2 y$, then $\frac{dy}{dx} = \frac{1}{\sec^2 y}$

We know:

$\sec^2 y = 1 + \tan^2 y$

Since $x = \tan y$

then $\sec^2 y = 1 + x^2$

Hence, $\frac{dy}{dx} = \frac{1}{1+x^2}$

$$\frac{d(\tan^{-1} y)}{dx} = \frac{1}{1 + x^2}$$

GENERALLY:

- $$\frac{d\left(\sin^{-1}\left(\frac{x}{a}\right)\right)}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$$
- $$\frac{d\left(\cos^{-1}\left(\frac{x}{a}\right)\right)}{dx} = -\frac{1}{\sqrt{a^2 - x^2}}$$
- $$\frac{d\left(\tan^{-1}\left(\frac{x}{a}\right)\right)}{dx} = \frac{a}{a^2 + x^2}$$

IMPLICIT DIFFERENTIATION

For functions that cannot be written in the form $y = f(x)$, we use implicit differentiation to obtain $\frac{dy}{dx}$.

We treat y as a function in terms of x.

Therefore, $\boldsymbol{\frac{d(y^n)}{dx} = n \cdot y^{n-1} \cdot \frac{dy}{dx}}$

Example : $\frac{d(y^3)}{dx} = 3 \cdot y^2 \cdot \frac{dy}{dx}$

PARTIAL DIFFERENTIATION

If we have a function with 2 or more variables, i.e. $f(x, y)$, we can find the derivative of $f(x, y)$ w.r.t. one variable while holding the others as constants.

- $\frac{\partial f}{\partial x}$ → Partial derivative of f w.r.t. x by holding y constant.
- $\frac{\partial f}{\partial y}$ → Partial derivative of f w.r.t. y by holding x constant.

Example:

If $f(x, y) = x^2 + 2x^2y^2 + 3y$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$\boldsymbol{\frac{\partial f}{\partial x}}$ **(w.r.t. $\boldsymbol{x}$)**

$= x^2 + (2y^2)x^2 + (3y)$

$= 2x + 2(2y^2)x + 0$

$= 2x + 4y^2x$

$\boldsymbol{\frac{\partial f}{\partial x}}$ **(w.r.t. $\boldsymbol{x}$)**

$= (x^2) + 2(2x^2)y^2 + 3y$

$= 0 + 2(2x^2)y + 3$

$= 4x^2y + 3$

SECOND ORDER PARTIAL DERIVATIVES

- $f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x^2} = \frac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial x}$

- $f_{yy} = (f_y)_y = \frac{\partial^2 f}{\partial y^2} = \frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial y}$

- $f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial y}$

- $f_{yx} = (f_y)_x = \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial x}$

MODULE 1 : INTEGRATION

PARTIAL FRACTIONS

Rules:

If the denominator is:

- Linear : $\frac{f(x)}{(ax+b)\,(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$

- Repeated Linear : $\frac{f(x)}{(ax+b)^2} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2}$

- Quadratic : $\frac{f(x)}{(ax^2+bx+c)} = \frac{Ax+B}{ax^2+bx+c}$

- Repeated Quadratic : $\frac{f(x)}{(ax^2+\ bx+c)^2} = \frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2}$

IMPROPER PARTIAL FRACTIONS

By looking at the highest power of x in the numerator and in the denominator, we can determine if the fraction is proper or improper.

$$Improper: \quad \text{Highest power of } x \text{ in the numerator} \geq \text{Highest power of } x \text{ in the denominator}$$

$$Proper: \quad \text{Highest power of } x \text{ in the numerator} < \text{Highest power of } x \text{ in the denominator}$$

To fix an improper fraction, we must turn it into a proper fraction by means of long division.

INTEGRATION REVIEW

- $y = x^n$

  $\int x^n \cdot dx = \frac{x^{n+1}}{n+1} + C \quad , \quad n \neq 1$

- $y = ax^n$

  $\int ax^n \cdot dx = \frac{ax^{n+1}}{n+1} + C \quad , \quad n \neq 1$

- $y = a$

  $\int a \cdot dx = ax + C$

- $y = (ax+b)^n$

  $\int (ax+b)^n \cdot dx = \frac{(ax+b)^{n+1}}{(n+1)\,(a)} + C \quad , \quad n \neq 1$

METHODS OF INTEGRATION

1. **Normal Integration**

   Example:

   $y = (ax+b)^n \quad , \quad n \neq 1$

   $\int (ax+b)^n \cdot dx = \frac{(ax+b)^{n+1}}{(n+1)\,(a)} + C$

2. ***ln***

   When $n = 1$, use the rule **:** $\int \frac{\boldsymbol{f'(x)}}{\boldsymbol{f(x)}} \cdot \boldsymbol{dx} = \mathbf{ln}|\boldsymbol{f(x)}|$

   Example:

   $\int \frac{5x}{2x-1} \cdot dx$

   Divide by 2 throughout

   $= \frac{5}{2} \int \frac{2x}{2x-1} \cdot dx$ ***N.B.*** *When the 2's cancel, we get back the original question*

   $= \frac{5}{2} \ln|2x-1| + C$

3. **Inverse Trigs**

Example:

$$\int \frac{1}{1+x^2} \cdot dx = \tan^{-1} x + C$$

4. **Integration by substitution**

Example: $\int \frac{3x}{\sqrt{x^2+1}} \cdot dx$

Let $u = x^2 + 1$ $\quad \therefore \frac{du}{dx} = 2x \rightarrow dx = \frac{du}{2x}$

$$= \int \frac{3\cancel{x}}{\sqrt{u}} \cdot \frac{du}{2\cancel{x}}$$

$$= \int \frac{3}{\sqrt{u}} \cdot \frac{du}{2}$$

$$= \frac{3}{2} \int \frac{1}{\sqrt{u}} \cdot du$$

$$= \frac{3}{2} \int u^{1/2} \cdot du$$

$$= \frac{3}{2} \cdot \frac{u^{1/2}}{1/2} + C$$

$$= 3u^{1/2} + C$$

$$= 3\sqrt{x^2+1} + C$$

INTEGRATION BY PARTS

If we have $U \cdot V$, then its differential is as follows:

$$\frac{d(U \cdot V)}{dx} = U\left(\frac{dv}{dx}\right) + V\left(\frac{du}{dx}\right)$$

Make $U\left(\frac{dv}{dx}\right)$ the subject of the formula:

$$U\left(\frac{dv}{dx}\right) = \frac{d(U \cdot V)}{dx} - V\left(\frac{du}{dx}\right)$$

If we were to integrate, the above formula now becomes:

$$\int U\left(\frac{dv}{\cancel{dx}}\right) \cdot \cancel{dx} = \int \frac{d(U \cdot V)}{\cancel{dx}} \cdot \cancel{dx} - \int V\left(\frac{du}{\cancel{dx}}\right) \cdot \cancel{dx}$$

$\therefore$ The formula for Integration by parts is :

$$\int U \cdot dv = U.V - \int V \cdot du$$

REDUCTION FORMULA

A reduction formula is used to help solve difficult integrals especially when they involve powers or repeated patterns.

Example:

Let $\boldsymbol{I_n = \int (\ln x)^n \cdot dx}$

$u = (\ln x)^n$ $\qquad\qquad dv = 1$

$du = (n)\left(\frac{1}{x}\right)(\ln x)^{n-1}$ $\qquad\qquad v = x$

$= \frac{n}{x} \cdot (\ln x)^{n-1}$

Use the integration by parts formula : $\int U \cdot\ dv = U.V - \int V \cdot du$

$$I_n = (\ln x)^n \cdot x - \int \cancel{x} \cdot \frac{n}{\cancel{x}} \cdot (\ln x)^{n-1} \cdot dx$$

$$I_n = x \cdot (\ln x)^n - n \int (\ln x)^{n-1} \cdot dx$$

$$I_n = x \cdot (\ln x)^n - n \cdot I_{n-1}$$

TRAPEZIUM RULE

$$\int_a^b f(x) \cdot dx = \frac{h}{2} \cdot [\, f(x_2) + f(x_n) + 2\,[f(x) + f(x_2) + \cdots + f(x_{n-1})\,]\,]$$

Where $h = \frac{b-a}{n}$