MODULE 2

ARITHMETIC PROGRESSION (A.P.) REVIEW

General Term : $\boldsymbol{T_n = a + (n-1)\, d}$

Where $a =$ first term

$d =$ common difference

Example :

Find the 5th term in the arithmetic progression given $a = 2$ and $d = 5$

5th term $\rightarrow$ $T_5 = 2 + (5-1)\,5$

$T_5 = 22$

$= \{2, 7, 12, 17, 22, \ldots, \infty\} \rightarrow \textit{divergent}$

GEOMETRIC PROGRESSION (G.P.) REVIEW

General Term : $\boldsymbol{T_n = ar^{n-1}}$

Where $a =$ first term

$r =$ common ratio

Example:

Find the 5th term in the geometric progression given $a = 2$ and $r = \frac{1}{2}$

5th term $\rightarrow$ $T_5 = (2)\left(\frac{1}{2}\right)^{5-1}$

$T_5 = \frac{1}{8}$

$= \left\{2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{\infty}\right\} \rightarrow \textit{convergent}$

N.B. *Some sequences always follow a pattern. They are called* ***Periodic Sequences.***

Example : $y = \cos\theta$ $\quad\quad \theta = 0, \frac{\pi}{2}, \pi, \ldots$

$= \{1, 0, -1, 0, 1, 0, -1, \ldots\} \rightarrow \textit{periodic sequence}$

MODULE 2 : SEQUENCES

A sequence is a set of numbers or elements that are generated from a particular relationship.

Limits can be used to determine whether a sequence is divergent or convergent.

A sequence is said to be convergent if as n approaches infinity, the sequence approaches the limit, l.

$$\lim_{n\to\infty} a_n = l$$

The three techniques used for simplifying limits are in this order:

1. Factorization
2. Divide by the highest power
3. L' Hopital

Example:

Determine if the sequence $a_n = \frac{n+1}{n}$ is convergent.

$a_n = \frac{n+1}{n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n}$

Divide by the highest power

$= \lim_{n\to\infty} \frac{n}{n} + \frac{1}{n}$

$= \lim_{n\to\infty} 1 + \frac{1}{n}$

$= 1 + \frac{1}{\infty}$

$= 1$

Using L' Hopital :

Differentiate numerator and denominator and solve.

- $\frac{d(n+1)}{dn} = 1 \rightarrow \ Numerator$
- $\frac{d(n)}{dn} = 1 \rightarrow \ Denominator$

$\rightarrow \frac{1}{1} = 1$, $\therefore$ the limit exists.

The limit exists, therefore the sequence $a_n = \frac{n+1}{n}$ is convergent.

N.B.

A Geometric Progression (G.P.) converges when $-1 < r < 1$.

$$\boldsymbol{S_\infty = \frac{a}{1-r}}$$

RECURRENCE RELATION

A recurrence relation occurs when a term is only obtained by knowing its predecessor.

Example :

$u_{r+1} = 2 + u_r \quad , \quad u_1 = 1$

So,

- u_2

  We can denote $r = 1$ in u_{r+1} since $r + 1 = 2$, $\therefore r = 1$

  $u = 2 + u_1$

  $u_2 = 2 + 1 = 3$

And

- u_3

  $u_3 = 2 + u_2$

  $u_2 = 2 + 3 = 5$

Therefore, $1 , 3 , 5$ is an example of a recurrence relation.

PROOF BY MATHEMATICAL INDUCTION FOR SEQUENCES

In order to fulfil P.M.I., 4 main steps must be completed:

1. Show it holds for $n = 1$
2. When $n = k$, assume true
3. Prove $n = k + 1$
4. By P.M.I. …

In sequences, the given recurrence relation is used in step 3.

Example:

Given $u_{n+1} = u_n + n + 1 \quad , \quad n \geq 1 \quad , \quad u_1 = 1$.
Use P.M.I. to show that $u_n = \frac{1}{2} n\,(n + 1)$.

1. **When $n = 1$**

$$u_1 = \frac{1}{2}\,(1)(1 + 1)$$

$$u_1 = \frac{1}{2}\,(2)$$

$$u_1 = 1 \qquad \therefore \textit{it holds}$$

2. **Assume true for $n = k$**

$$u_k = \frac{1}{2}\,k\,(k + 1)$$

3. **Show it holds for $n = k + 1$**

$$u_{k+1} = \frac{1}{2}\,(k + 1)\,(k + 2) \quad \leftarrow \quad \textit{Prove this}$$

We know,

$$u_{k+1} = u_k + k + 1 \qquad (\textit{Replace } u_k \textit{ using step } 2)$$

$$u_{k+1} = \frac{1}{2}\,k\,(k + 1)\ + (k + 1)$$

$$u_{k+1} = \frac{1}{2}\,k\,(k + 1) + \frac{2}{2}\,(k + 1)$$

$$u_{k+1} = \frac{1}{2}\,(k + 1)\,[\,k + 2\,]$$

$$u_{k+1} = \frac{1}{2}\,(k + 1)\,(k + 2)$$

□

Since it holds for $n = 1,\ n = k$ and $n = k + 1$, then by P.M.I. it holds $\forall\, n \geq 1$.

MODULE 2 : SERIES

PROOF BY MATHEMATICAL INDUCTION FOR SERIES

Example:

Use P.M.I. to show $\sum_{r=1}^{n} \ r\,(r+1) = \frac{1}{3} \cdot n\,(n+1)(n+2)$, $\forall\, n \in \mathbb{N}$.

1. **When $\boldsymbol{n = 1}$**

LHS	**RHS**
$1(1+1) = 2$	$\frac{1}{3}\,(1)\,(1+1)\,(1+2)$
	$\frac{1}{3}\,(1)\,(2)\,(3) = 2$

$\therefore\ it\ holds$

2. **Assume true for $\boldsymbol{n = k}$**

$$\sum_{r=1}^{k} \ r\,(r+1) = \frac{1}{3} \cdot k\,(k+1)(k+2)$$

3. **Show it holds for $\boldsymbol{n = k+1}$**

$$\sum_{r=1}^{k+1} \ r\,(r+1) = \frac{1}{3} \cdot (k+1)\,(k+2)(k+3) \quad \leftarrow \quad Prove\ this$$

Since,

$$\sum_{r=1}^{k+1} \ r\,(r+1) = \sum_{r=1}^{k} \ r\,(r+1) + (k+1)\ term$$

Then, $\boldsymbol{(Replace\ \sum_{r=1}^{k} r\,(r+1)\ \ using\ step\ 2)}$

$$\sum_{r=1}^{k+1} \ r\,(r+1) = \left[\ \frac{1}{3} \cdot \ k\,(k+1)(k+2)\right] + [(k+1)(k+2)]$$

$$= \frac{1}{3} \cdot \ k\,(k+1)\,(k+2) + \frac{3}{3}\,(k+1)\,(k+2)$$

$$= \frac{1}{3}\,(k+1)\,(k+2)\,[k+3]$$

$$= \frac{1}{3}\,(k+1)\,(k+2)\,(k+3)$$

□

Since it holds for $n = 1,\ n = k$ and $n = k+1$, then by P.M.I. it holds $\forall\, n \in \mathbb{N}$.

A series is a sum of all the terms in a sequence.

N.B. $\sum$ *Indicates series*

$$\sum_{r=1}^{n} a_r = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

Two popular types of series are:

1. Arithmetic Series
2. Geometric Series

- **ARITHMETIC SERIES**

$$\boldsymbol{S_n = \frac{n}{2}\,(2a + (n-1)\,d)}$$

General Term : $\boldsymbol{T_n = a + (n-1)\,d}$

Where a = first term

d = common difference

- **GEOMETRIC SERIES**

$$\boldsymbol{S_n = \frac{a\,(1 - r^n)}{1 - r}}$$

General Term : $\boldsymbol{T_n = ar^{n-1}}$

Where a = first term

r = common ratio

Example :

1. Given $S_n = n(3 - 2n)$

    (i) Determine if A.P. or G.P.

    (ii) n^{th} term

(i) $1^{st}\ term, a = S_1 = (1)\ (3 - 2(1))$

$= (1)\ (1) = 1$

$2^{nd}\ term,\ T_2 = S_2 - S_1$

$= [\ 2\ (3 - 2\ (2)\ )] - 1$

$= 2(-1) - 1 = -3$

$3^{rd}\ term,\ T_3 = S_3 - S_2$

$= [\ 3\ (3 - 2\ (3)\ )] - [2\ (3 - 2\ (2)\ )]$

$= 3\ (-3) - 2\ (-1)$

$= -9 + 2 = -7$

Subtract two consecutive terms twice to find the common difference , d, if it is the same then it is A.P.

$d = -3 - 1 = -4$

$d = -7 - (-3) = -4$

$\therefore a = 1 \qquad d = -4$

$\therefore A.P.$

(ii) $n^{th} term,\ T_n = a + (n - 1)\ d$

$T_n = 1 + (n - 1)(-4)$

Example :

2. Given $S_n = 2^{n+3} - 8$
   (i) Determine if A.P. or G.P.
   (ii) n^{th} term

(i) $1^{st}\ term, a = S_1 = 2^{1+3} - 8$

$$= 2^4 - 8 = 8$$

$2^{nd}\ term,\ T_2 = S_2 - S_1$

$$= (2^{2+3} - 8) - 8$$

$$= 24 - 8 = 16$$

$3^{rd}\ term,\ T_3 = S_3 - S_2$

$$= (2^{3+3} - 8) - (2^{2+3} - 8)$$

$$= 56 - 24 = 32$$

Divide two consecutive terms twice to find the common ratio, r, if it is the same then it is G.P.

$r = \frac{16}{8} = 2$

$r = \frac{32}{16} = 2$

$\therefore a = 8 \qquad r = 2$
$\therefore G.P.$

(ii) $n^{th} term,\ T_n = ar^{n-1}$

$$T_n = 8\ (2)^{n-1}$$

If we sum all the terms in a sequence, a_n, we will obtain a series.

$$\sum_{all} a_n$$

<u>RULES FOR SERIES</u>

$$\sum_{r=1}^{n} k = kn$$

$$\sum_{r=1}^{n} r = \frac{1}{2} \cdot n\,(n+1) \quad \leftarrow \quad \textit{In Formula Sheet}$$

$$\sum_{r=1}^{n} (a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$$

$$\sum_{r=1}^{n} k a_r = k \sum_{r=1}^{n} a_r$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6} \cdot n\,(n+1)\,(2n+1) \quad \leftarrow \quad \textit{In Formula Sheet}$$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4} \cdot n^2\,(n+1)^2 \quad \leftarrow \quad \textit{In Formula Sheet}$$

$$\sum_{r=1}^{n} r = \sum_{r=1}^{p} r + \sum_{r=p+1}^{n} r$$

METHOD OF DIFFERENCES

To find the summation of $g(x)$, i.e. $\sum g(x)$ and there is a way to write it in terms of two different functions, for example: $\sum\ (f(x_1) - f(x_2))$.
The summation can be found by expanding the functions into the 1st term, 2nd term to the nth term.

Method of Differences allows for all the terms besides the smallest term (not necessarily the first) and the largest term (not necessarily the last) to be cancelled off so that the summation can be found.

Example:

$\sum_{r=1}^{n} \frac{1}{r(r+1)}$ ← Break up into Partial Fractions

So,

$$\frac{1}{r\,(r+1)} = \frac{A}{r} + \frac{B}{r+1}$$

$\times\ (r(r+1))$

$1 = A(r+1) + B(r)$

$1 = Ar + A + Br$

Compare Coefficients

- $A + B = 0$
- $\boldsymbol{A = 1}$

Since $A = 1$ then

$A + B = 0$

$1 + B = 0$

$\boldsymbol{B = -1}$

$\therefore \frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$

So,

$$\sum_{r=1}^{n} \frac{1}{r\,(r+1)} = \sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1}\right)$$

And

1st, 2nd, 3rd, $(n-1)$ term, (n) term

$$\sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+1}\right) = \left[\frac{1}{1} - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{1}{3}\right] + \left[\frac{1}{3} - \frac{1}{4}\right] + \cdots + \left[\frac{1}{n-1} - \frac{1}{n}\right] + \left[\frac{1}{n} - \frac{1}{n+1}\right]$$

Remove Brackets

$= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1}$

$= 1 - \frac{1}{n+1}$

N.B.

$$\sum_{r=1}^{\infty} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1}$$
$$= 1 - \frac{1}{\infty}$$
$$= 1$$

This shows that Method of Differences is a technique used to show convergence.

CONVERGENCE OF A SERIES

If $S_n \sum_{r=1}^{n} a_r$ converges then it implies that as n approaches infinity, then the series approaches/ converges to the limit, l.

$$\lim_{n\to\infty} S_n = l$$

There are two (2) methods to show that a series is convergent :

Method 1 : Method of Differences

Once we can show $\lim_{n\to\infty} S_n = l$, then S_n is convergent, else S_n is divergent.

Method 2 :

If series $\sum_{n=1}^{n} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

This means that the SEQUENCE converges to 0.

Else, if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{n} a_n$ is divergent.

Example:

Is $\sum_{n=1}^{n} \left(\frac{2n}{2n+3}\right)$ convergent?

$\downarrow$

Sequence

$$\lim_{n\to\infty} \left(\frac{2n}{2n+3}\right) = \lim_{n\to\infty} \frac{\frac{2n}{n}}{\frac{2n}{n} + \frac{3}{n}}$$
$$= \lim_{n\to\infty} \frac{2}{2 + \frac{3}{n}}$$

As n approaches ∞

$$= \frac{2}{2 + \frac{3}{\infty}} = \frac{2}{2+0} = \frac{2}{2} = 1$$

Since $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{n} \left(\frac{2n}{2n+3}\right)$ is divergent.

MODULE 2 : POWER SERIES

BINOMIAL EXPANSION

Formula:

$$(\boldsymbol{a}+\boldsymbol{b})^{\boldsymbol{n}} = \boldsymbol{a}^{\boldsymbol{n}} + \binom{\boldsymbol{n}}{\boldsymbol{1}} \boldsymbol{a}^{\boldsymbol{n-1}} \cdot \boldsymbol{b} + \binom{\boldsymbol{n}}{\boldsymbol{2}} \boldsymbol{a}^{\boldsymbol{n-2}} \cdot \boldsymbol{b}^{\boldsymbol{2}} + \cdots + \binom{\boldsymbol{n}}{\boldsymbol{n}} \boldsymbol{b}^{\boldsymbol{n}}$$

N.B. *Binomial expansion is used when* n *is a whole number.*

When n *is negative for a fraction, binomial expansion cannot be used.*

Example:

$$(3 - x^2)^{4} \leftarrow n$$
$$\downarrow \quad \downarrow$$
$$a \quad b$$

$$(3 - x^2)^4 = 3^4 + \binom{4}{1} 3^{4-1} \cdot (-x^2)^1 + \binom{4}{2} 3^{4-2} \cdot (-x^2)^2 + \binom{4}{3} 3^{4-3} \cdot (-x^2)^3 + \binom{4}{4} (-x^2)^4$$

$$= 81 + (4)\,(3^3)\,(-x^2) + (6)\,(3^2)\,(x^4) + (4)\,(3)\,(-x^6) + x^8$$

$$= 81 - 108x^2 + 54x^4 - 12x^6 + x^8$$

The General Term $\left[\binom{\boldsymbol{n}}{\boldsymbol{r}} \boldsymbol{a}^{\boldsymbol{n-r}} \cdot \boldsymbol{b}^{\boldsymbol{r}}\right]$ can be used to obtain a specific coefficient for a particular power without having to go through the entire binomial expansion.

Example:

Obtain the coefficient of x^6 from $(3 - x^2)^4$

We know : $a = 3 \qquad b = -x^2 \qquad n = 4$

Put into General Term Formula

$$\binom{4}{r} (3)^{4-r} (-x^2)^r$$

When $r = 3$, x^6 is obtained.

$$\therefore \binom{4}{3} (3)^{4-3} (-x^2)^3 = (4)\,(3)\,(-x^6)$$

$$= -12x^6$$

Therefore, the coefficient of x^6 is -12 .

N.B. ^nC_r *follows Pascal's triangle.*

Therefore, it cannot handle negative $(-)$ *or rational* $\left(\frac{a}{b}\right)$ *numbers.*

If n is negative, $n \leq 0$, or n is a rational number, $n \in \frac{a}{b}$, then:

$$(a+b)^n = a^n + \left(\frac{n}{1!}\right) a^{n-1} \cdot b + \underbrace{\left(\frac{(n)(n-1)}{2!}\right)}_{\substack{\uparrow \\ \textit{Product of 2} \\ \textit{consecutive terms}}} a^{n-2} \cdot b^2 + \underbrace{\left(\frac{(n)(n-1)(n-2)}{3!}\right)}_{\substack{\uparrow \\ \textit{Product of 3} \\ \textit{consecutive terms}}} a^{n-3} \cdot b^3 + \cdots$$

General Term :

$$\binom{n}{r} = \frac{(n)\,(n-1)\,(n-2) \ldots (n-(r-1))}{r!} \quad \leftarrow \quad \textit{In Formula Sheet}$$

Example:

Obtain the first 4 terms in the expansion of $(1-x)^{\frac{1}{2}}$

$a = 1 \qquad b = -x \qquad n = \frac{1}{2}$

$$= (1)^{\frac{1}{2}} + \left(\frac{\frac{1}{2}}{1!}\right)(1)^{\frac{1}{2}-1} \cdot (-x) + \left(\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}\right)(1)^{\frac{1}{2}-2} \cdot (-x)^2 + \left(\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\right)(1)^{\frac{1}{2}-3} \cdot (-x)^3 + \cdots$$

$$= 1 + \left(\frac{1}{2}\right)(1)(-x) + \left(-\frac{1}{8}\right)(1)(x^2) + \left(\frac{1}{16}\right)(1)(-x^3) + \cdots$$

$$= 1 - \frac{1}{2} \cdot x - \frac{1}{8} \cdot x^2 - \frac{1}{16} \cdot x^3 + \cdots$$

$$= \underbrace{1}_{\substack{\uparrow \\ 1^{st} \\ \text{term}}} \underbrace{- \frac{1}{2} \cdot x}_{\substack{\uparrow \\ 2^{nd} \\ \text{term}}} \underbrace{- \frac{1}{8} \cdot x^2}_{\substack{\uparrow \\ 3^{rd} \\ \text{term}}} \underbrace{- \frac{1}{16} \cdot x^3}_{\substack{\uparrow \\ 4^{th} \\ \text{term}}} + \cdots$$

TAYLOR SERIES

$$f(x) = f(a) + (x-a)\cdot f'(a) + \frac{(x-a)^2}{2!}\cdot f''(a) + \frac{(x-a)^3}{3!}\cdot f'''(a) + \cdots$$

Where : 1. $f(x)\ is\ differentiable$

2. $f(x)\ is\ defined\ around\ x = a$

Example:

Find the first four (4) non-zero terms of $f(x) = \sqrt{x}$ about $x = 9$

- $f(x) = \sqrt{x}$, $f(9) = \sqrt{9} = 3$
- $f'(x) = \frac{1}{2}\, x^{-\frac{1}{2}}$, $f'(9) = \frac{1}{2}\,(9)^{-\frac{1}{2}} = \frac{1}{6}$
- $f''(x) = -\frac{1}{4}\, x^{-\frac{3}{2}}$, $f''(9) = -\frac{1}{4}\,(9)^{-\frac{3}{2}} = -\frac{1}{108}$
- $f'''(x) = \frac{3}{8}\, x^{-\frac{5}{2}}$, $f'''(9) = \frac{3}{8}\,(9)^{-\frac{5}{2}} = \frac{1}{648}$

$$\therefore f(x) = 3 + (x-9)\left(\frac{1}{6}\right) + \frac{(x-9)^2}{2!}\left(-\frac{1}{108}\right) + \frac{(x-9)^3}{3!}\left(\frac{1}{648}\right) + \cdots$$

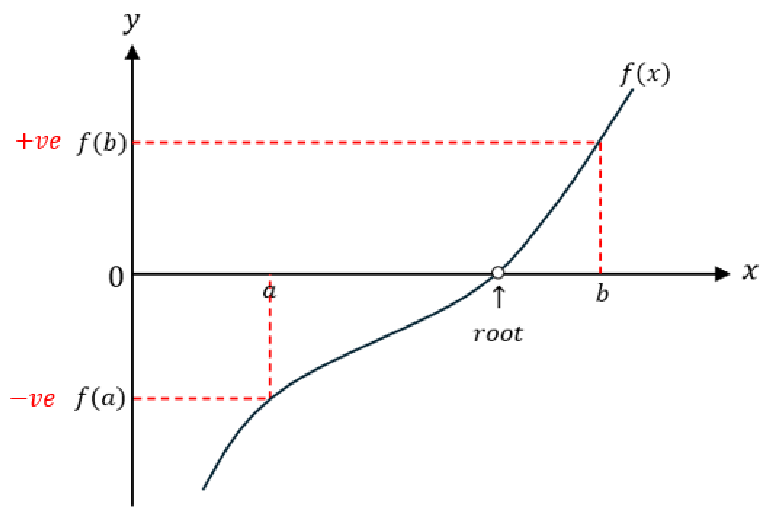
$$f(x) = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2 + \frac{1}{3888}(x-9)^3 + \cdots$$

MACLAURIN'S EXPANSION

$$f(x) = f(0) + (x)\cdot f'(0) + \frac{x^2}{2!}\cdot f''(0) + \frac{x^3}{3!}\cdot f'''(0) + \cdots$$

N.B. *For Maclaurin's Expansion, the same formula is used however* $a = 0$

MODULE 2 : ROOTS OF AN EQUATION



To establish the existence of a root, (i.e. where the function cuts the x-axis) we use Intermediate Value Theorem (I.V.T.)

<u>INTERMEDIATE VALUE THEOREM</u>

If $y = f(x)$ is a continuous function in the interval $a < x < b$ and a root exists between the limits a and b, we can find $f(a)$ and $f(b)$ and their product.

The product of $f(a)$ and $f(b)$ should result in a negative value.

i.e. $\boldsymbol{f(a) \cdot f(b) < 0}$

A negative value or a sign change indicates that at least one root exists.

To ensure that only 1 root exists between a and b . We have to show that the function is strictly increasing or strictly decreasing. This is done by differentiating the function and show if it is more or less than zero.

- $Strictly\ increasing : f'(x) > 0$
- $Strictly\ decreasing : f'(x) < 0$

Some techniques used to estimate a root when $y = 0$ are:

- Interval Bisection
- Linear Interpolation
- Iterative Approach
- Newton-Raphson Method

Example:

Determine if a root exists between 2 and 3 for the function $f(x) = x^3 - 25$.

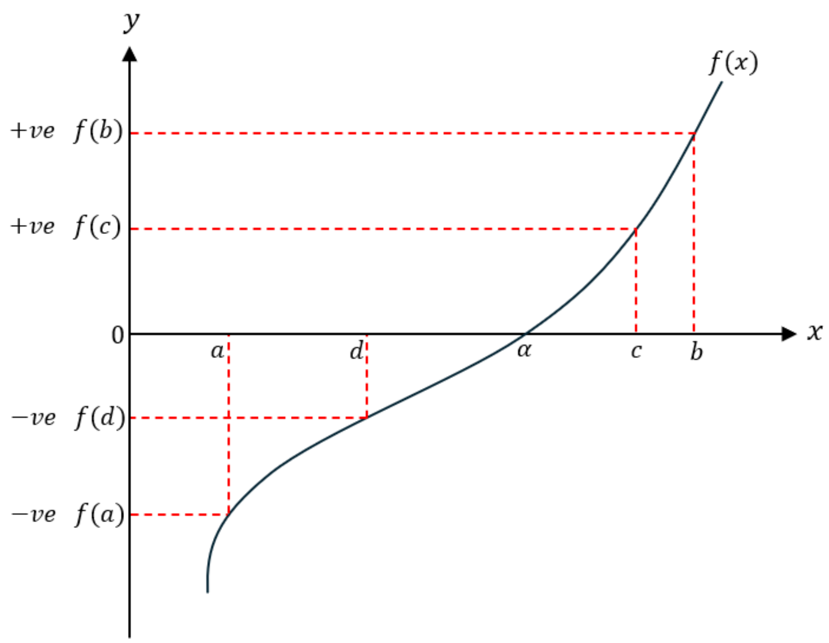
$x^3 - 25 = 0$

- $f(2) = 2^3 - 25 = -17$
- $f(3) = 3^3 - 25 = 2$

$f(2) \cdot f(3) = -34$ $\quad\quad\quad$ $-34 < 0$

∴ By I.V.T. , at least one root exists.

INTERVAL BISECTION



- $c = \frac{a+b}{2}$
- $f(c) = +ve$
- c is closer to α, this allows us to narrow the boundaries.
- The new boundaries are $[a\,, c]$

Explanation:

The new boundaries allow I.V.T. to hold, since $f(a)$ is $-ve$ and $f(c)$ is $+ve$.

This means that $f(c) \cdot f(a) < 0$.

We do not choose $f(c)$ and $f(b)$ since α is not between those points and also because it does not satisfy I.V.T.

ALWAYS choose boundaries that satisfy the Intermediate Value Theorem.

This technique is repeated until $[a\,, c]$ are the same to some specific decimal place.

Note:

If $f(x) = 0$ has a root, then to estimate this root between a given interval $[a, b]$, we can :

1. **Find the midpoint of $[a, b]$ i.e. $X_m = \frac{a+b}{2}$.**
2. **Find $f(X_m)$.**
3. **By using Intermediate Value Theorem (I.V.T.), determine the new interval.**

N.B. Repeat until the root is found to a specific degree of accuracy, i.e. 2 d.p. or 3 d.p.

Example:

(i) Use the intermediate value theorem to show that $f(x) = \sqrt{x} - \cos x$ has a root in the interval $[0, 1]$.

- $f(0) = \sqrt{0} - \cos 0 = -1$
- $f(1) = \sqrt{1} - \cos 1 = 0.46$

Since, $f(0) \cdot f(1) < 0$, by I.V.T. a root exists.

(ii) Use two iterations of the interval bisection method approximate the root of f in the interval $[0, 1]$.

$f(x) = \sqrt{x} - \cos x \qquad [0, 1]$

- $f(0) = -1$
- $f(1) = 0.46$

<u>1st Iteration</u>

$\frac{0+1}{2} = 0.5 \quad , \qquad f(0.5) = -0.17$

New Boundaries $[0.5 , 1]$

<u>2nd Iteration</u>

$\frac{0.5+1}{2} = 0.75 \quad , \qquad f(0.75) = 0.13$

New Boundaries $[0.5 , 0.75]$

LINEAR INTERPOLATION

Like Interval Bisection, the Intermediate Value Theorem must always hold.

However, the formula it follows is not the same.

The formula for Linear Interpolation is:

$$\boldsymbol{x_1} = \frac{\boldsymbol{b}\,|\boldsymbol{f(a)}| \;+\; \boldsymbol{a}\,|\boldsymbol{f(b)}|}{|\boldsymbol{f(b)}| \;+\; |\boldsymbol{f(a)}|}$$

Where I.V.T. must hold, i.e. $f(x_1) \cdot f(a) < 0$

Example:

(i) Use the intermediate value theorem to show that the equation $4\cos x - x^3 + 2 = 0$ has a root in the interval $[1, 1.5]$.

$f(x) = 4\cos x - x^3 + 2 \qquad [1, 1.5]$

- $f(1) = 4\cos(1) - (1)^3 + 2 = 3.161$
- $f(1.5) = 4\cos(1.5) - (1.5)^3 + 2 = -1.092$

Since, $f(1) \cdot f(1.5) < 0$, by I.V.T. a root exists in the interval $[1, 1.5]$.

(ii) Use linear interpolation to approximate the value of the root of the equation $4\cos x - x^3 + 2 = 0$ in the interval $[1, 1.5]$, correct to two decimal places.

1st Iteration, x_1

modulus, use $+ve$

$$\frac{(1.5)(3.161) + (1)(\overset{\downarrow}{+}1.092)}{(1.092) + (3.161)} = 1.372$$

$f(1.372) = 4\cos(1.372) - (1.372)^3 + 2 = 0.207$

New Boundaries $[1.372 \;,\; 1.5]$

2nd Iteration, x_2

$$\frac{(1.5)(0.207)+(1.372)(1.092)}{(1.092)+(0.207)}=1.392$$

$f(1.392) = 4\cos(1.392) - (1.392)^3 + 2 = 0.0142$

New Boundaries [1.392 , 1.5]

3rd Iteration, x_3

$$\frac{(1.5)(0.0142)+(1.392)(1.092)}{(1.092)+(0.0142)}=\mathbf{1.393}$$

$f(1.393) = 4\cos(1.393) - (1.393)^3 + 2 = 0.0044$

New Boundaries [1.393 , 1.5]

4th Iteration, x_4

$$\frac{(1.5)(0.0044)+(1.393)(1.092)}{(1.092)+(0.0044)}=\mathbf{1.393}$$

$f(1.393) = 4\cos(1.393) - (1.393)^3 + 2 = 0.0044$

∴ **It converges, indicating that the root is approximately 1.393.**

NEWTON-RAPHSON METHOD

Newton-Raphson Method provides the first estimate, x_1 .

To find x_2 , we use the formula:

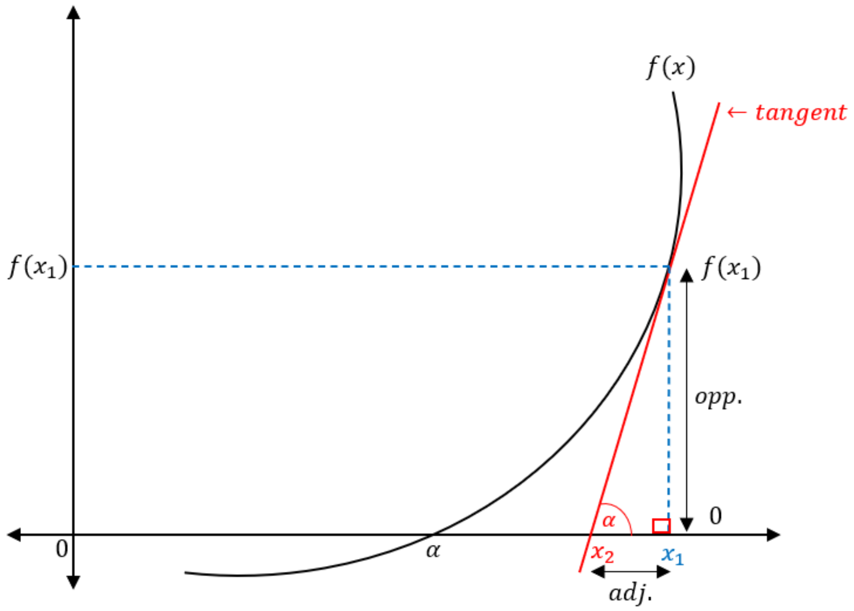
$$\boldsymbol{x_2 = x_1 - \frac{f(x_1)}{f'(x_2)}}$$

Newton-Raphson is a Convergence Method.

The General formula for the Newton-Raphson Method is as follows:

$$\boldsymbol{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Proof for the Newton-Raphson Method formula:



When given x_1 , we can find $f(x_1)$. Since x_1 is far from α, a tangent is drawn at the point where x_1 meets the curve. Where the tangent cuts the $x - axis$, a new estimate is obtained (x_2). The tangent then makes an angle α, with the $x - axis$.

We know : $\tan\alpha = \dfrac{opposite}{adjecent}$

$\therefore \tan\alpha = \dfrac{f(x_1) - 0}{x_1 - x_2}$ $\begin{matrix} \leftarrow \textit{Length of opposite} \\ \leftarrow \textit{Length of adjacent} \end{matrix}$

Remember : $\tan\alpha$ represents the slope

We know : Slope is gradient which is $\dfrac{dy}{dx}$

$\therefore$ Slope of the line at $x_1 : f'(x_1)$

$$f'(x_1) = \frac{f(x_1)}{x_1 - x_2}$$

Make x_2 subject of the formula

$\rightarrow \quad (f'(x_1))(x_1 - x_2) = f(x_1)$

$\rightarrow \quad x_1 - x_2 = \dfrac{f(x_1)}{f'(x_1)}$

$\rightarrow \quad x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

The above formula represents: $\boldsymbol{x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}}$ $\square$

Example:

The equation $3e^x = 1 - 2\ln x$ is known to have a root in the interval $[0, 1]$.

Taking $x_1 = 0.2$ as the first approximation of the root, use the Newton-Raphson Method to find a second approximation, x_2, of the root in the interval $[0, 1]$.

$3e^x = 1 - 2\ln x \quad \rightarrow \quad 3e^x - 1 + 2\ln x = 0$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

- $f(x) = 3e^x - 1 + 2\ln x$

  $\therefore f(0.2) = 3e^{0.2} - 1 + 2\ln(0.2) = -0.555$

- $f'(x) = 3e^x + \frac{2}{x}$

  $\therefore f'(0.2) = 3e^{0.2} + \frac{2}{0.2} = 13.664$

$$\therefore x_2 = 0.2 - \left(\frac{-0.555}{13.664}\right) = 0.241$$

Continue for x_3, x_4, ... until the same decimal is obtained twice. That decimal represents the root approximation.